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On Hamiltonian Dynamics

by

A.R. Walker

Thesis submitted for the degree of Doctor of Philosophy
at the University of Warwick.

1970

On Hamiltonian Dynamics

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A.R.Walker

Paper 1 : On Hamiltonian Dynamics.

Paper 2 : Morse-Smale type Inequalities for
Hamiltonian Dynamical Systems.

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Introduction

In this paper we are primarily concerned with the problem of classification of \mathbb{E} -equivalence (energy equivalence) classes of Hamiltonian Dynamical Systems, and the problem of finding a complete set of invariants for each equivalence class. In fact we will find such a classification for an open dense set of Hamiltonian Dynamical Systems on a compact manifold with boundary, and for an open set of Hamiltonian Dynamical Systems on a non-compact manifold. Thus most of the work will be formulated in terms of Hamiltonian Dynamics (and hence the frequent prefix 'Hamiltonian' in our definitions). However the techniques and results developed below also have relevance to the theory of functions on manifolds and we will point out these corollaries wherever possible.

In chapter 1 we define a manifold with singularities and develop the theory of tubular neighbourhoods for a manifold with singularities embedded in a manifold with boundary. The concept of a manifold with singularities is basic to this work. The results developed in

chapter 1 are in parallel to the results for tubular neighbourhoods of a submanifold in a manifold with boundary and will be used extensively in the later chapters. The techniques used follow in spirit those of Lang [4], to which work, with [1], we refer the reader for the basic results on manifolds used in this chapter.

In chapter 2 we state the definitions and standard results which we will need later, rephrasing some of them to suit our approach. We begin with a survey of functions on manifolds and of Hamiltonian dynamics. For basic information on these topics we refer the reader to [3], and [6],[7] resp. Finally in chapter 2 we define a graph and prove the results we will need later. So far as we know, these results (essentially on the ordering of vertices in an oriented graph) are new.

In chapter 3 we define a Hamiltonian foliation on a compact manifold with boundary. A Hamiltonian foliation is a generalisation of a non-degenerate function on a manifold with boundary, and as such we use it to define the graph associated to a Hamiltonian foliation and a Complete Set of Invariants (C.S.I.) for a Hamiltonian foliation. Both of these are invariants of \mathbb{R} -equivalence classes of Hamiltonian foliations, and as the name suggests the C.S.I. of two Hamiltonian

foliations are identical if and only if the Hamiltonian foliations are \mathbb{E} -equivalent. We apply the results so obtained to Hamiltonian dynamical systems and functions to find

- a) a C.S.I. for an \mathbb{E} -equivalence class of Hamiltonian dynamical systems (or functions),
- b) a graph for a Hamiltonian dynamical system (or function),
- c) various criteria for a C.S.I. to be a C.S.I. for a function or Hamiltonian dynamical system, and for a locally Hamiltonian dynamical system to be \mathbb{E} -equivalent to, to be \mathbb{H} -equivalent to [6], or to be a (globally) Hamiltonian dynamical system.

Finally we look at the set of stable Hamiltonian dynamical systems and functions (v.i.) and show that they are in fact \mathbb{E} -stable.

In chapter 4 we apply the results of chapter 3 to Hamiltonian dynamical systems and functions on compact 2-manifolds with boundary. We show that the results of chapter 3 can be considerably strengthened. In particular we show that the stable Hamiltonian dynamical systems are \mathbb{H} -stable, that the \mathbb{H} -equivalence classes can be classified by the graphs associated to the systems, and that we can find a classification of the systems on a given 2-manifold with boundary. We develop parallel results for functions on a 2-manifold with boundary.

The results of chapter 4 are complementary to those of Peixoto (as yet unpublished) in which he uses graph techniques to classify Morse-Smale systems on a 2-manifold. The result that \mathcal{H} -stable systems are open dense on a 2-manifold is complementary to the result of Robinson [11] which states that \mathcal{H} -stable systems are not open dense in dimension ≥ 4 .

In chapter 5 we give a brief outline of the generalisation of the results of chapter 3 to non-compact manifolds. We show that there is an open set of \mathcal{E} -stable functions defined by

- a) all closed energy surfaces are compact,
- b) all critical points are non-degenerate,
- c) each closed energy surface contains at most one critical point.

We generalise the graph and the C.S.I. and the related results. Finally we look at the functions on a cotangent bundle T^*M of the form $H = T + V$ where T is a kinetic energy function and V a potential function, and determine necessary and sufficient conditions for such a function to be \mathcal{E} -stable.

I should like to thank my supervisor, Peter Walters, for a great deal of help and encouragement in the work for this paper.

Chapter 1

Manifolds with Singularities

In this chapter we develop the concept of a manifold with singularities. The motivation, which will become apparent later, comes from studying the inverse image of a critical value of a real valued function on a manifold.

We begin by defining a manifold with singularities. We then define a submanifold with singularities in a manifold with boundary and prove existence and uniqueness of tubular neighbourhoods for this situation. Finally we investigate some special functions on these tubular neighbourhoods.

1.1 Definition

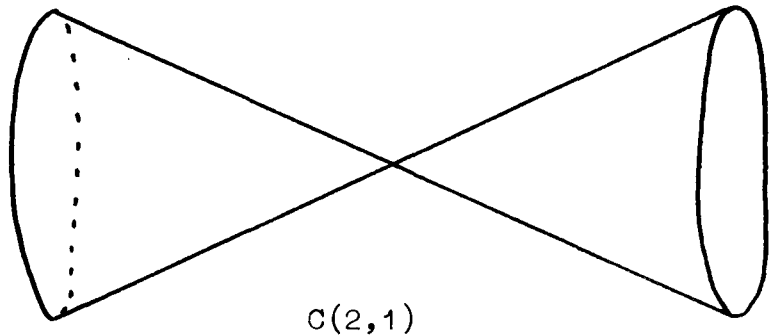
Let \mathbb{R}^n denote Euclidean n -space, \mathbb{R}_+^n the half space $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. Define $C(n, \lambda)$ to be the set

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1^2 + \dots + x_\lambda^2) - (x_{\lambda+1}^2 + \dots + x_{n+1}^2) = 0\} \subset \mathbb{R}^{n+1}, \text{ and } C(n, \lambda, m) = C(n, \lambda) \times \mathbb{R}^m.$$

Let T be a topological space. An open set $U \subset T$ is said to be locally (n-)Euclidean if for all $x \in U$ there exists a neighbourhood U_x of x and a homeomorphism of U_x onto a neighbourhood of $0 \in \mathbb{R}^n$. A point $x \in T$ is said to be (n-)Euclidean if it has a locally (n-)Euclidean neighbourhood.

A point $x \in T$ is said to be half-(n-)Euclidean if there exists a neighbourhood U_x of x and a homeomorphism h from U_x to a neighbourhood of $0 \in \mathbb{R}_+^n$ with $h(x) = 0$.

A point $x \in T$ is said to be (n-)singular if there exists a neighbourhood U_x of x and a homeomorphism h of U_x onto a neighbourhood of $0 \in C(m, \lambda, n-m) \subset \mathbb{R}^{n+1}$ with $h(x) = 0$. Note that $0 \leq \lambda \leq m+1$ and $m \leq n$. Also $C(m, \lambda, n-m)$ is isomorphic to $C(m, m-\lambda+1, n-m)$. Let $\mu = \min(\lambda, m-\lambda+1)$; then $0 \leq \mu \leq [\frac{m+1}{2}]$ and (m, μ) , or



μ if $m = n$, is called the singularity index of x . We will primarily be interested in the case $m = n$, $\mu \neq 0$. In the case $\mu = 0$, $C(m, \lambda, n-m) \cong \mathbb{R}^{n-m}$ and there is no 'singularity'. Hence, in the following, we shall assume that $\mu \neq 0$.

An n -manifold-with-boundary-and-singularities (n -manifold w.b.a.s.), M^n , is a paracompact Hausdorff space in which every point is either n -Euclidean, half n -Euclidean, or n -singular.

The set of Euclidean points of M is called the interior of M , denoted $\overset{\circ}{M}$. The set of half-Euclidean points of M is called the boundary of M , denoted ∂M , and the set of n -singular points of M is called the singular set of M , denoted $S(M)$. These sets are mutually exclusive and exhaustive of M , and ∂M and $S(M)$ are closed subsets of M .

If $S(M) = \emptyset$ we call M a manifold-with-boundary (manifold w.b.); if $\partial M = \emptyset$, we call M a manifold-with-singularities (manifold w.s.); if $S(M) = \partial M = \emptyset$ we call M a manifold.

n is called the dimension of M .

A smooth differentiable structure on M , a manifold w.b.a.s., is a collection Φ of real valued functions, each defined in an open subset of M , such that

1. if $U \subset V$ and $f \in \Phi$ is defined on V , then $f|_U \in \Phi$.
2. if $U = \bigcup U_\alpha$ and f is defined on U with $f|_{U_\alpha} \in \Phi$ for each α , then $f \in \Phi$.
3. every point $x \in M$ has a neighbourhood U and a homeomorphism h of U onto an open neighbourhood of 0 in \mathbb{R}^n or \mathbb{R}_+^n or $C(m, \lambda, n-m)$ such that a function f defined on $V \subset U$ lies in Φ if and only if $f \circ h^{-1}$ is a smooth function on $h(V)$.

Remarks 1. If U is an open neighbourhood of $0 \in \mathbb{R}_+^n$ and $f: U \rightarrow \mathbb{R}$, we say f is a smooth function if there exists an open neighbourhood V of $0 \in \mathbb{R}^n$ containing U , and a smooth function $f': V \rightarrow \mathbb{R}$ extending f to V . If g' is any other smooth function extending f , then $f = g$ on U and all their derivatives agree on U . Thus we can define uniquely the derivatives of f at all points of U .

2. Similarly if U is an open neighbourhood of $0 \in C(m, \lambda, n-m)$ and $f: U \rightarrow \mathbb{R}$, we say f is a smooth function if there exists an open neighbourhood V of $0 \in \mathbb{R}^{n+}$ containing U and a smooth function $f': V \rightarrow \mathbb{R}$ extending f . If g' is any other smooth function extending f then, as above, $f' = g'$ agree on U and all their derivatives in directions tangent to $C(m, \lambda, n-m)$ agree on U .

A smooth n -manifold w.b.a.s. is an n -manifold w.b.a.s. and a smooth differentiable structure on it. We now drop the adjective 'smooth', i.e. 'manifold w.b.a.s.' will in

future mean 'smooth manifold w.b.a.s.'

An open set $W \subset M$ and a homeomorphism h having property (3) above relative to the differentiable structure on M are called a coordinate neighbourhood and coordinate map respectively. (W, h) is called a coordinate chart. If $m \in M$ and $h(m) = (x_1(m), \dots, x_n(m)) \in \mathbb{R}^n$ or \mathbb{R}_+^n or $h(m) = (x_1(m), \dots, x_{n+1}(m)) \in C(m, \lambda, n-m)$, then the functions x_i are called local coordinates.

A map $f: M \longrightarrow M'$ of manifolds w.b.a.s. is smooth if for all $h \in \Phi(M')$, $h \circ f \in \Phi(M)$. A 1-1 correspondence $f: M \longrightarrow M'$ is a diffeomorphism if both f and f^{-1} are smooth. A map $f: M \longrightarrow M'$ is almost smooth if for all $h \in \Phi(M')$ with h defined on U and $U \cap S(M') = \emptyset$, we have $h \circ f \in \Phi(M)$. i.e. f is smooth except on $S(M')$.

1.2 Submanifolds

Let $m \leq n$. Then we have a natural embedding of \mathbb{R}^m in \mathbb{R}^n , viz. $\mathbb{R}^m = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{m+1} = \dots = x_n = 0\}$. Let $(\mathbb{R}^n, \mathbb{R}^m)$ denote \mathbb{R}^n with \mathbb{R}^m embedded in it as above. Similarly we have

$$(\mathbb{R}_+^n, \mathbb{R}_+^m) : \quad \mathbb{R}_+^m \subset \mathbb{R}_+^n, \quad m \leq n, \quad \text{and} \quad \partial \mathbb{R}_+^m \cap \mathbb{R}_+^n \subset \partial \mathbb{R}_+^n \\ \mathring{\mathbb{R}}_+^m \cap \mathbb{R}_+^n \subset \mathring{\mathbb{R}}_+^n.$$

$$(\mathbb{R}^n, \mathbb{R}_+^m) : \mathbb{R}_+^m \subset \mathbb{R}^n, m \leq n.$$

$$(\mathbb{R}_+^n, \mathbb{R}^m) : \mathbb{R}^m \subset \mathbb{R}_+^n, m \leq n, \text{ and } \mathbb{R}^m \subset \partial \mathbb{R}_+^n.$$

$$(\mathbb{R}^n, C(q, \lambda, m-q)) : C(q, \lambda, m-q) = C(q, \lambda) \times \mathbb{R}^{m-q} \\ \subset \mathbb{R}^{q+1} \times \mathbb{R}^{m-q} = \mathbb{R}^{m+1} \subset \mathbb{R}^n, m < n.$$

$$(C(p, \mu, n-p), C(q, \lambda, m-q)) : C(q, \lambda) \subset C(p, \mu), \\ \mathbb{R}^{m-q} \subset \mathbb{R}^{n-p}, m-q \leq n-p, q \leq p, \text{ and} \\ \lambda \leq \mu \text{ or } q-\lambda+1 \leq \mu.$$

Let M be a subset of a manifold w.b.a.s. N^n . M is a submanifold w.b.a.s. of N , of dimension m , codimension $n-m$, if for each point $x \in M$ there exists a coordinate chart (U, h) for N , containing x , such that

$$h: (U, U \cap M) \longrightarrow (P, Q),$$

where (P, Q) is one of the six pairs defined above, and $M \cap \partial N = \overline{M} \cap \partial N$.

If M is a closed subset of N then M is called a closed submanifold w.b.a.s. M has a natural structure of manifold w.b.a.s. with differentiable structure defined by the restriction to M of the structure on N . We call this the induced structure on M . Note that the existence of coordinate charts is guaranteed by the definition.

M is called a submanifold (resp. submanifold w.b., submanifold w.s.) if M with the induced structure is a manifold (resp. manifold w.b., manifold w.s.).

Note that if Q is a component of ∂M then $Q \subset \partial M$

or $Q \subset \overset{\circ}{N}$, and $\overset{\circ}{M} \cup S(M) \subset \overset{\circ}{N} \cup S(N)$.

A map $f: M \longrightarrow N$ of manifolds w.b.a.s. is called an embedding if $f(M)$ is a submanifold w.b.a.s. of N and f is a diffeomorphism of M with $f(M)$.

A map $f: M \longrightarrow N$ is called an almost smooth embedding if f is an almost smooth map and a homeomorphism of M with $f(M)$.

1.3 Neighbourhood for a singular point

We can now define a smooth vector bundle over M ([1], [2], [4], [5]). Note that if M is a manifold w.b.a.s. (resp. manifold w.b., manifold w.s., manifold) then a vector bundle over M has a natural structure of manifold w.b.a.s. (resp. manifold w.b., manifold w.s., manifold) derived from the local product structure. In particular if M is a manifold w.b. we can define tensor bundles over M and so prove existence and uniqueness theorems for tubular neighbourhoods of submanifolds of M , [1], [4], and for tubular neighbourhoods of ∂M , [1].

Our immediate objective now will be to prove existence and uniqueness theorems for tubular neighbourhoods in the special case when M^{n-1} is a submanifold w.s. of codimension one in a manifold w.b. N^n , whose singularities are all of type $C(n-1, \lambda)$. This is the case in which we will be interested later.

In this case $S(M)$ will be a union of isolated points. Below 'submanifold w.s.' will denote a submanifold w.s. of this type unless specifically stated to be otherwise.

The definitions and results below follow in spirit those of Lang [4]. Wall [1] uses stronger definitions which restrict his results to the compact case.

We conjecture that the results proved for tubular neighbourhoods of the special submanifolds w.s. hold also for submanifolds w.s. in general. Certainly all the necessary structures generalise easily, and, if we keep the restriction that $S(M)$ is a union of isolated points, all the proofs go through with trivial modifications to the case of submanifolds w.s. of higher codimension. However the proofs break down when we allow singularities of type $C(q, \lambda, m)$, $m \neq 0$, and more sophisticated techniques appear to be necessary.

We begin by defining a vector bundle structure for a neighbourhood of a singular point.

Define $D(n, \lambda)$, the complement of $C(n, \lambda)$, to be the set $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1^2 + \dots + x_\lambda^2)(x_{\lambda+1}^2 + \dots + x_{n+1}^2) = 0\} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = \dots = x_\lambda = 0 \text{ or } x_{\lambda+1} = \dots = x_{n+1} = 0\}$. Thus $D(n, \lambda)$ consists of two complementary subspaces \mathbb{R}^λ and \mathbb{R}^μ , $\mu = n+1-\lambda$, of \mathbb{R}^{n+1} . We will usually denote \mathbb{R}^λ and \mathbb{R}^μ by D^λ and D^μ since we wish to emphasize their disc structure (v.i.) rather than

their linear space structure. We write $D(n, \lambda) = D^\lambda \cup D^\mu = \mathbb{R}^\lambda \cup \mathbb{R}^\mu$ with the convention that D^λ and D^μ meet only in $\{0\}$. Let $S^{\lambda-1}$ be the unit sphere in \mathbb{R}^λ . Then we have a 'fibration' of D^λ as $S^{\lambda-1} \times \mathbb{R}_+ / S^{\lambda-1} \times \{0\}$ with fibres $\{(t^{\frac{1}{2}}x_1, \dots, t^{\frac{1}{2}}x_\lambda) : t \in \mathbb{R}_+\}$ for each $(x_1, \dots, x_\lambda) \in S^{\lambda-1}$. (Here as elsewhere we use the term 'fibration' in an intuitive rather than a rigorous sense.) Each fibre has the natural linear structure of \mathbb{R}_+ given by the parameter t , and all fibres have a common zero. The linear structure is clearly independent of the sphere about $\{0\}$ used to define the unit vector on each fibre. We call the above structure on D^λ , with the natural topology, a disc structure on D^λ . We take a similar disc structure on D^μ , differing only in the fact that we parametrize the fibres by the non-positive real numbers, \mathbb{R}_- . Now D^λ and D^μ , each with its disc structure, define a disc structure on $D(n, \lambda)$. D^λ and D^μ are called the component spaces of $D(n, \lambda)$.

Let $f: D^\lambda \longrightarrow D^{\lambda'}$. We say f is fibre linear if f is a smooth fibre preserving map and is linear on each fibre. Clearly $f(0) = 0$, for $0 \in D^\lambda$ is also the zero of each fibre. We say $f: D(n, \lambda) \longrightarrow D(n', \lambda')$ is fibre linear if it is fibre linear on each component space of $D(n, \lambda)$, and maps $D^\lambda \longrightarrow D^{\lambda'}$ and $D^\mu \longrightarrow D^{\mu'}$.

Let $F(D(n,\lambda), D(n',\lambda'))$ denote the set of fibre linear maps $f: D(n,\lambda) \longrightarrow D(n',\lambda')$. Note that the composition of two fibre linear maps is fibre linear.

We say $\| \cdot \| : D(n,\lambda) \longrightarrow \mathbb{R}$ is a norm on $D(n,\lambda)$ if its restriction to each fibre of $D(n,\lambda)$ is a norm on that fibre.

Let $z \in C(n,\lambda) - \{0\} \subset \mathbb{R}^{n+1}$. Define the fibre through z to be the set of points

$$\{(\mu z_1, \dots, \mu z_\lambda, \frac{1}{\mu} z_{\lambda+1}, \dots, \frac{1}{\mu} z_{n+1}) \in \mathbb{R}^{n+1} : 0 < \mu < \infty,$$

$$z = (z_1, \dots, z_{n+1})\}.$$

The fibre through z is parametrized by

$$t = \mu^2(z_1^2 + \dots + z_\lambda^2) - \frac{1}{\mu^2}(z_{\lambda+1}^2 + \dots + z_{n+1}^2), \text{ and we denote}$$

by t_z the point on the fibre through z with parameter t . Thus $0_z = z$.

Let U be a neighbourhood of 0 in $C(n,\lambda)$. Define the associated neighbourhood $A(U) \subset \mathbb{R}^{n+1}$ of U to be the set of points $D(n,\lambda) \cup \{t_z : t \in \mathbb{R}, z \in U - \{0\}\}$. $A(U)$ has a fibration with fibres $D(n,\lambda)$ over 0 and \mathbb{R} elsewhere. $D(n,\lambda)$ itself has a fibration defined by its disc structure.

The above will be taken as the standard fibration over a neighbourhood of 0 in $C(n,\lambda)$, and will be used below to construct a singular bundle for a submanifold w.s.

We will denote a point in $A(U)$ by (z, v) or v_z where $z \in U - 0$ and v is a point in the fibre over z . Note that $A(U)$ is not a product, despite this notation. In each fibre we denote multiplication by a scalar (w.r.t. the natural linear structure) by $t(z, v) = (z, tv)$, and if (z, u) and (z, v) are in the same fibre for $z \neq 0$ or in the same fibre of the fibre for $z = 0$, we can define addition by $(z, u) + (z, v) = (z, u+v)$.

To each point in $A(U)$ there is associated a real parameter t defined as above. Thus we have a map $q : A(U) \longrightarrow \mathbb{R} : (x_1, \dots, x_{n+1}) \longmapsto (x_1^2 + \dots + x_\lambda^2) - (x_{\lambda+1}^2 + \dots + x_{n+1}^2)$ which associates to each point $x \in A(U)$ its parameter t . Clearly q is a smooth map. The map $|q| : A(U) \longrightarrow \mathbb{R}_+ : (z, v) \longmapsto |q(z, v)|$ defines a smooth norm on $A(U)$, i.e. $|q|$ restricted to each fibre is a norm on the fibre.

Define $A_\varepsilon(U) = \{(z, v) \in A(U) : |q(z, v)| < \varepsilon\}$, for $\varepsilon > 0$. If U is an open neighbourhood of 0 in $C(n, \lambda)$, then $A(U)$ is an open neighbourhood of $D(n, \lambda)$ in \mathbb{R}^{n+1} , and $A_\varepsilon(U)$ is an open neighbourhood of U in $A(U)$.

We can also define the associated neighbourhood $A(U)$ when U is a subset of $C(n, \lambda)$ not containing 0 .

Let $x \in A(U)$. Then we have a (continuous) pro-

jection $p : A(U) \longrightarrow U$ defined by $p(x) = 0$ if $x \in D(n, \lambda)$, $p(t_z) = z$ if $z \in C(n, \lambda) - \{0\}$.

1.4 Singular Bundles

Let M^n be a manifold w.s. with $S(M)$ a union of isolated points.

A singular bundle over M is a triple (π, E, M) - where E is a topological space, $\pi : E \longrightarrow M$ a surjective map - and a cover $\{(U_i, h_i)\}$ of M by coordinate charts, such that

1. if U_i is locally Euclidean then there exists a homeomorphism $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}$ so that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{R} \\ \pi \searrow & & \swarrow \pi_1 \\ & U_i & \end{array} \quad \text{commutes.}$$

(π_1 denotes projection on the first factor)

Also, writing ϕ_{ix} for $\phi_i|_{\pi^{-1}(x)}$, if $x \in U_i \cap U_j$ then

- a) $\phi_{ij}(x) = \phi_{ix} \circ \phi_{jx}^{-1} \in L(\mathbb{R}, \mathbb{R})$,
- b) $\phi_{ij} : U_i \cap U_j \longrightarrow L(\mathbb{R}, \mathbb{R})$
 $\quad \quad \quad : x \rightsquigarrow \phi_{ij}(x) \text{ is smooth,}$

and

2. if U_i contains a singular point y , there exists a homeomorphism $\phi_i : \pi^{-1}(U_i) \longrightarrow A(U_i)$ (strictly $A(h_i(U_i))$), but for simplicity we will suppress the

homeomorphism $h_i : U_i \longrightarrow C(n, \lambda)$) such that

$$\begin{array}{ccc}
 \pi^{-1}(U_i) & \xrightarrow{\varphi_i} & A(U_i) \\
 \pi \searrow & & \swarrow p \\
 & U_i &
 \end{array}
 \quad \text{commutes}$$

and, if $x \in (U_i \cap U_j) - S(M)$ then

a) $\varphi_{ij}(x) \in L(\mathbb{R}, \mathbb{R})$,

b) $\varphi_{ij} : (U_i \cap U_j) - S(M) \longrightarrow L(\mathbb{R}, \mathbb{R})$
is smooth,

if $y \in U_i \cap U_j \cap S(M)$ then

c) $\varphi_{ij}(y) \in F(D(n, \lambda), D(n, \lambda))$,

d) $\varphi_i \circ \varphi_j^{-1} : A(U_j) \longrightarrow A(U_i)$ is smooth in
a neighbourhood of y .

Remarks 1. Strictly we should define a singular bundle to be a triple (π, E, M) and a maximal cover of M by coordinate neighbourhoods satisfying (1) and (2).

2. The maps φ_i are called trivialisations and the maps φ_{ij} are called transit functions.

3. We can give E a smooth manifold structure w.r.t. which the φ_i are diffeomorphisms and each fibre $\underline{E_x} = \pi^{-1}(x)$ has the natural linear structure of \mathbb{R} or $D(n, \lambda)$.

4. $(\pi, E - \pi^{-1}(S(M)), M - S(M))$ is a vector bundle.

A singular bundle map between singular bundles (π, E, M) and (π', E', M') is a pair of smooth maps (f, \bar{f}) such that

- a) $f: E \longrightarrow E'$ and $\bar{f}: M \longrightarrow M'$,
 b) $\bar{f}(S(M)) \subset S(M')$ and $\bar{f}(\overset{\circ}{M}) \subset \overset{\circ}{M}'$,

c)
$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\bar{f}} & M' \end{array} \quad \text{commutes,}$$

d) the induced map $f_x : E_x \longrightarrow E'_{\bar{f}(x)}$ is linear or fibre linear according as $x \in \overset{\circ}{M}$ or $x \in S(M)$.

(f, \bar{f}) is a singular bundle isomorphism if there exists a singular bundle map (g, \bar{g}) such that $f \circ g = \text{id}(E')$ and $g \circ f = \text{id}(E)$.

Suppose that (π, E, M) is a singular bundle over M . We define $O(E)$ (or $O(\pi)$) : $M \longrightarrow E$ to be the map $x \longmapsto 0_x \in E_x$ for all x in M . Then $O(E)$ embeds M in E as a submanifold w.s. We call $O(E)$ the zero section of (π, E, M) , but note that general sections are not defined. Whenever convenient we will identify M with its image $O(E)(M) \subset E$, which we will also (by abuse of language) call the zero section and denote by $O(E)$.

Theorem 1 Let $\{U_i\}$ be a cover of a manifold w.s. M (with $S(M)$ a set of isolated points) by coordinate neighbourhoods and let

$$\varphi_{ij} : (U_i \cap U_j) \cap S(M) \longrightarrow L(\mathbb{R}, \mathbb{R}) \quad \text{and}$$

$$\varphi_{ij} : U_i \cap U_j \cap S(M) \longrightarrow F(D(n, \lambda), D(n, \lambda))$$

be smooth maps such that

$$\varphi_{ij}(x) \circ \varphi_{jk}(x) = \varphi_{ik}(x) \text{ for all } x \in U_i \cap U_j \cap U_k,$$

and $\varphi_{ij} : A(U_j) \longrightarrow A(U_i) : (z, v) \longmapsto \varphi_{ij}(z) \cdot v$

is smooth in a neighbourhood of $A(U_j) \cap S(M)$.

Then there exists a singular bundle (π, E, M) over M with transit functions φ_{ij} .

Proof If U_i is a locally Euclidean coordinate neighbourhood, define $V_i = U_i \times \mathbb{R}$. If U_i contains a singular point, define $V_i = A(U_i)$. Let $E' = \bigcup_i V_i = \{(i, z, v) : z \in U_i, v \text{ is in the fibre of } z \text{ in } V_i\}$. Define an equivalence relation \sim on E' by $(i, y, u) \sim (j, z, v)$ if and only if $y = z$ and $\varphi_{ji}(z) \cdot u = v$.

Let $E = E' / \sim$. The natural projection $E' \longrightarrow M$ induces a well-defined projection $\pi : E \longrightarrow M$.

$\varphi_{ii} = 1$, so $(i, y, u) \sim (i, z, v)$ if and only if $y = z$ and $u = v$. Thus

$$\varphi_i : \pi^{-1}(U_i) \longrightarrow V_i : \{(i, z, v)\} \sim \longmapsto (z, v)$$

is well defined and bijective, and $\pi = \pi_i \circ \varphi_i$ (or $\pi = p \circ \varphi_i$).

Finally define $\varphi_{ix} : \pi^{-1}(x) \longrightarrow \mathbb{R}$ or $D(n, \lambda)$ by $\{(i, z, v)\} \sim \longmapsto v$. This is well defined, and if $x \in U_i \cap U_j$ then

$$\begin{aligned}\varphi_{jx} \circ \varphi_{ix}^{-1}(v) &= \varphi_{jx}\{(i, x, v)\} \sim = \varphi_{jx}\{(j, x, \varphi_{ji}(x) \cdot v)\} \sim \\ &= \varphi_{ji}(x) \cdot v.\end{aligned}$$

Thus we have a bundle (π, E, M) with transit functions φ_{ij} . ||

Lemma 2 If (π, E, M) is a singular bundle over M then there exists a smooth norm on E , i.e. a map $\| \cdot \| : E \longrightarrow \mathbb{R}$, continuous on E and smooth on $E - O(E)$, such that $\| \cdot \|_x = \| \cdot \|$ restricted to $\pi^{-1}(x)$ is a norm for the fibre over x .

Proof Choose a cover of M by coordinate neighbourhoods U_i such that over each U_i there exists a trivialisation $\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}$ or $A(U_i)$. Choose a partition of unity $\{\theta_i\}$ subordinate to $\{U_i\}$.

We first note that if U_i is locally Euclidean then there exists a smooth norm on $U_i \times \mathbb{R}$, viz. $\|(x, t)\| = |t|$. Similarly if U_i contains a singular point, there exists a smooth norm on $A(U_i)$, viz.

$\|(x, v)\| = |s|$, where $x \in C(n, \lambda)$, $p(x, v) = x$, and s is the parameter associated to v in the fibre

over x . (Note that the set of points with associated parameter t or $-t$ is given by

$$|x_1^2 + \dots + x_\lambda^2 - x_{\lambda+1}^2 - \dots - x_{n+1}^2| = |t|, \text{ hence the}$$

smoothness of $\| \cdot \|$.)

Now for $v_x \in E$ define

$$\|v_x\| = \sum_i \theta_i(x) \|\varphi_i(v_x)\|$$

This is well defined and continuous on E and smooth on $E - O(E)$. On each fibre we have

$$\|v\|_x = \sum_i \theta_i(x) \|\varphi_{ix}(v)\|$$

for $v \in \pi^{-1}(x)$. The verification that $\|\cdot\|_x$ is a norm on the fibre over x is now trivial. \parallel

Lemma 3 With the norm on E defined above we have for each $x \in M$ a neighbourhood U_x of x and a trivialisation φ for π , over U_x , and constants $k_x, K_x > 0$ such that for all $v \in \pi^{-1}(U_x)$

$$k_x \|\varphi(v)\| \leq \|v\| \leq K_x \|\varphi(v)\|.$$

Proof We have $\|v_x\| = \sum_i \theta_i(x) \|\varphi_i(v_x)\|$. For each x there exists j, k_x, K_x, U_x , such that for all $y \in U_x$

$$1. \theta_j(y) \geq k_x > 0.$$

$$2. U_x \text{ meets precisely } N \text{ of the } U_i.$$

$$3. \|\varphi_{ij}(y)\| \leq K_x/N \text{ for all } y \in U_x \cap U_j \cap U_i.$$

$$\text{Then } k_x \|\varphi_j(v)\| \leq \|v\| \leq \sum_i \theta_i(\pi v) \|\varphi_i(v)\|$$

$$\leq \sum_i \theta_i(\pi v) \|\varphi_{ij}(\pi v)\| \cdot \|\varphi_j(v)\|$$

$$\leq K_x \|\varphi_j(v)\|$$

for all $v \in E$ such that $\pi(v) \in U_x$. \parallel

1.5 Existence of Tubular Neighbourhoods

We now have sufficient structure to define and prove the existence of tubular neighbourhoods.

Let M^{n-1} be a submanifold w.s. of a manifold w.b. N . A tubular neighbourhood of M in N consists of a singular bundle (π, E, M) over M , an open neighbourhood Z of the zero section $O(E)$ in E , and a diffeomorphism $f : Z \longrightarrow U$ of Z onto an open set $U \subset N$ containing M , such that

$$\begin{array}{ccc}
 & Z & \\
 O(E) \uparrow & \searrow f & \\
 M & \xrightarrow{\subset} & N
 \end{array}
 \quad \text{commutes.}$$

If the above conditions hold when Z is a closed neighbourhood of the zero section in E and a submanifold w.b. of E , and U a closed neighbourhood of M in N , then we say we have a closed tubular neighbourhood of M in N . Clearly the existence of a closed tubular neighbourhood implies the existence of a tubular neighbourhood, and if M is compact then the converse holds also.

Theorem 1 Let M^{n-1} be a closed submanifold w.s. of a manifold w.b. N^n . Then there exists a tubular neighbourhood of M in N .

Sketch of proof The idea of the proof is as follows:
 We show first that there exists an almost smooth fibre preserving embedding of a singular bundle E in $TN/M = F$, such that if $x \in \overset{\circ}{M}$ and outside a chosen neighbourhood of $S(M)$, then the embedded fibre E_x is tangent to the normal fibre $N_x \overset{\circ}{M}$ at 0 in F_x . Now any spray on N induces an exponential map $\exp : TN \longrightarrow N$. Thus we have induced exponential maps $F \longrightarrow N$ and therefore $E \longrightarrow N$. To show existence of a tubular neighbourhood it is sufficient to show that $\exp : E \longrightarrow N$ is a local diffeomorphism on $0(E)$. At each $x \in M$ where E_x is tangent to a fibre of the normal bundle this follows as in the proof of the existence theorem for submanifolds, and if x is in our chosen neighbourhood of $S(M)$, we show that we can choose a local spray making \exp a local diffeomorphism. We now choose local sprays arbitrarily elsewhere, and form a global spray by partition of unity techniques. this gives the required result.

Proof M^{n-1} is a closed submanifold w.s. of the manifold w.b. N^n . Let (τ, TN, N) be the tangent bundle of N , (τ', F, M) , $F = TN/M$, the restriction to M of the tangent bundle. Then F is a (general) manifold w.s.. $\overset{\circ}{M}$ is a submanifold of N , so we have a normal bundle of $\overset{\circ}{M}$ in N , $(n, N\overset{\circ}{M}, \overset{\circ}{M})$. We may

suppose that $\overset{\circ}{NM} \subset F \subset TN$.

$S(M)$ is a set of isolated singular points $\{y_i\}$. Choose disjoint coordinate neighbourhoods $U_i \subset M$ with $y_i \in U_i$ for each i . In each U_i choose smaller coordinate neighbourhoods of y_i , U_i^1, U_i^2, U_i^3 , with $\bar{U}_i^3 \subset U_i^2$, $\bar{U}_i^2 \subset U_i^1$, $\bar{U}_i^1 \subset U_i$. Let $V^j = U_i U_i^j$, for $j = 1, 2, 3$, and $V = U_i U_i$. Choose a smooth function $t: M \longrightarrow \mathbb{R}$ such that $t(x) = 1$ if $x \in \bar{V}^2$, and $t(x) = 0$ if $x \notin V^1$, and $t(x) \in (0, 1)$ if $x \in V^1 - \bar{V}^2$.

Choose a cover of $M - V$ by coordinate neighbourhoods $U_j \subset M - \bar{V}^1$. The U_j together with the U_i already chosen give a cover of M by coordinate neighbourhoods subordinate to the cover $\{V, M - \bar{V}^1\}$. W.l.o.g. we may assume that there does not exist an index k so that $U_k \subset V - \bar{V}^1$.

For each $U_j \subset M - \bar{V}^1$ we may suppose that U_j has been chosen so that there exists a trivialisation

$$\phi_j : \tau'^{-1}(U_j) \longrightarrow U_j \times \mathbb{R}^n : v_x \longmapsto (x; v_1, \dots, v_n)$$

so that if v_x is tangential to M at x , then

$$v_x \longmapsto (x; v_1, \dots, v_{n-1}, 0), \text{ and if } v_x \in N_x \overset{\circ}{M} \text{ then}$$

$$v_x \longmapsto (x; 0, \dots, 0, v_n). \text{ For let } U'_j \text{ be a coordinate}$$

neighbourhood in N with $U_j = U'_j \cap M$ such that

(U'_j, U_j) is homeomorphic to a neighbourhood of 0 in $(\mathbb{R}^n, \mathbb{R}^{n-1})$.

Take coordinates (x_1, \dots, x_n) in \mathbb{R}^n and basis $\partial/\partial x_1, \dots, \partial/\partial x_n$ for the tangent space over each point. Then the first condition is satisfied and we have an embedding of the normal fibres as $(x; t) \rightsquigarrow (x; a_1 t, \dots, a_n t)$ where $a_i = a_i(x)$ and the normal fibre is transversal to the first $n-1$ coordinates. i.e. $a_n(x) \neq 0$. Thus we can make a smooth change of coordinates $\partial/\partial x'_i = \partial/\partial x_i$ for $i = 1, \dots, n-1$, and $\partial/\partial x'_n = \sum a_i \partial/\partial x_i$, on each tangent space to achieve the required trivialisation.

Similarly for each $U_i \subset V$ we may suppose that $U_i \subset C(n-1, \lambda)$ is chosen so that there exists a trivialisation $\phi_i : \tau^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ for which

1. if v_x is tangential to M at $x \in S(M)$ then $v_x \rightsquigarrow (x; v_1, \dots, v_n)$ where $(v_1, \dots, v_n) \in C(n-1, \lambda)$
2. if v_x is tangential to M at $x \in \overset{\circ}{M}$ then $v_x \rightsquigarrow (x; v_1, \dots, v_n)$ where $x = (x_1, \dots, x_n) \in C(n-1, \lambda)$ and $v_1 x_1 + \dots + v_\lambda x_\lambda - v_{\lambda+1} x_{\lambda+1} - \dots - v_n x_n = 0$
3. if $x \notin U_i^3$ and $v_x \in N_x \overset{\circ}{M}$ then $v_x \rightsquigarrow (x; v_1, \dots, v_n)$ where $x \in C(n-1, \lambda)$ and $v_1 = kx_1, \dots, v_\lambda = kx_\lambda, v_{\lambda+1} = -kx_{\lambda+1}, \dots, v_n = -kx_n$ for some $k \in \mathbb{R}$.

For we again choose canonical coordinates for $U_i \subset \mathbb{C}^n$ and take $\partial/\partial x_1, \dots, \partial/\partial x_n$ as basis for the tangent space over each point. Then the first two conditions

are satisfied and $N_{\overset{\circ}{x}}M$ is transversal to the plane given by $v_1x_1 + \dots + v_\lambda x_\lambda - v_{\lambda+1}x_{\lambda+1} - \dots - v_n x_n = 0$.

As before we can make a smooth coordinate change which gives the normal fibre in the required form on $U_i - U_i^3$.

Now if $U_j \subset M - \bar{V}^1$ define

$$\chi_j : U_j \times \mathbb{R} \longrightarrow U_j \times \mathbb{R}^n : (x; t) \rightsquigarrow (x; 0, \dots, 0, t)$$

$$\text{and } \psi_j : U_j \times \mathbb{R} \longrightarrow \tau'^{-1}(U_j) \text{ by } \psi_j = \phi_j^{-1} \circ \chi_j.$$

If $U_i \subset V$ define $\chi_i : A(U_i) \longrightarrow U_i \times \mathbb{R}^n$:

$$x \rightsquigarrow (p(x); t(p(x))(x - p(x)) + (1 - t(p(x)))f(x))$$

where t is the real valued function defined above,

and $f(x) = 0 \in \mathbb{R}^n$ if $x \in D(n, \lambda) \subset A(U_i)$ and

$f(s_z) = (sz_1, \dots, sz_\lambda, -sz_{\lambda+1}, \dots, -sz_n)$ when $s_z = x \in A(U_i)$ and $z \in U_i - \{0\}$.

Define $\psi_i : A(U_i) \longrightarrow \tau'^{-1}(U_i)$ to be $\phi_i^{-1} \circ \chi_i$.

For each $x \in U_k$ define $\psi_{kx} = \psi_k|_{\text{(fibre over } x \text{)}}$.

Then for each $x \in U_k \cap U_1$ we can define maps

$$\psi_{k1}(x) = \psi_{kx}^{-1} \circ \psi_{1x} \in L(\mathbb{R}, \mathbb{R}) \text{ and } \psi_{k1} : U_k \cap U_1$$

$$\longrightarrow L(\mathbb{R}, \mathbb{R}) : x \rightsquigarrow \psi_{k1}(x). \text{ These maps are well}$$

defined, for each map ψ_{kx} , with $x \in U_k \cap U_1$,

is a linear injection into the normal fibre over

x in F . Thus we have defined a singular bundle

(π, E, M) over M using the neighbourhoods U_k and

the maps ψ_{k1} . The maps ψ_k define an embedding of

E in F . This embedding, ψ , is almost smooth since $p(x)$ is smooth except on $D(n, \lambda)$ and is fibre preserving.

We see from the definitions that ψ embeds the fibres E_x along those of the normal bundle when $x \in M - V^1$ and tangent at 0 to those of the normal bundle when $x \notin V^3$. Thus, as in the corresponding proof for submanifolds [Lang, 4, p. 74], given any spray γ on N with induced exponential map $\exp : E \longrightarrow N$, \exp is a local diffeomorphism at each $x \in M - \bar{V}^3$.

We now construct such a spray with the additional properties that it induces a local diffeomorphism on V^3 also. Extend each coordinate neighbourhood $U_k \subset M$ to a coordinate neighbourhood $U'_k \subset N$ with $U'_k \cap M = U_k$. Extend these neighbourhoods to a cover of N by additional coordinate neighbourhoods $\{U'_m\}$ disjoint from M . Choose a partition of unity $\{\theta_j\}$ on N subordinate to this cover, and in particular for each U'_i containing a singular point y_i choose θ_i so that $\theta_i(x) = 1$ if $x \in \bar{U}_i^2$, $\theta_i(x) = 0$ if $x \in M - U_i^1$, $\theta_i(x) \in (0, 1)$ if $x \in U_i^1 - \bar{U}_i^2$. On each $U'_i \subset N$ containing y_i choose a trivial local spray, i.e.

$$\begin{aligned} \gamma_i : U_i \times \mathbb{R}^n &\longrightarrow U_i \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\ &: (x, v) \rightsquigarrow (x, v, v, 0) . \end{aligned}$$

and choose local sprays arbitrarily elsewhere. Piece

these local sprays together using the partition of unity $\{\theta_i\}$ to form a global spray γ . γ induces an exponential map $\exp : TN \longrightarrow N$ which induces in turn the map $\exp \circ \psi : E \longrightarrow N$. It follows that $\exp \circ \psi$ is a local diffeomorphism at each $x \in M - \bar{V}^3 \subset E$.

Define $U_i^{2\bullet}$ to be the set $\theta_i^{-1}(1) \subset N$. Then $U_i^{2\bullet} \cap M = U_i^2$. On each $U_i^{2\bullet}$ the global spray γ is trivial by construction. Suppose that $(x, v) \in U_i^{2\bullet} \times \mathbb{R}^n$, and that the integral curve through (x, v) is given by $\phi(t) = (y(t), w(t))$. Then

$$\frac{d\phi}{dt} = \left(\frac{dy}{dt}, \frac{dw}{dt} \right) = (w, 0).$$

Hence $w(t) = w(0) = v$, and $y(t) = y(0) + vt = x + vt$. Thus in a neighbourhood of $U_i^{2\bullet} \times \{0\}$ \exp has the form $(x, v) \rightsquigarrow x + v : TN \longrightarrow N$. In a neighbourhood of $U_i^2 \subset M$, $\exp \circ \psi : E \longrightarrow N$ has the form $x \rightsquigarrow (p(x), x - p(x)) \rightsquigarrow x$, i.e. $\exp \circ \psi$ is smooth and a local diffeomorphism.

We have shown that $\exp \circ \psi : E \longrightarrow N$ is a local diffeomorphism on the zero section. To show that there exists an open neighbourhood of the zero section on which $\exp \circ \psi$ is a global diffeomorphism is now easy, and in fact is precisely the

proof given by Lang [4,p.74] for the submanifold case. ||

1.6 Uniqueness of Tubular Neighbourhoods

Let M be a manifold w.s. with $S(M)$ a set of isolated singular points.

A singular bundle (π, E, M) over M is said to be compressible if, given an open neighbourhood Z of the zero section, there exists a diffeomorphism $\phi : E \longrightarrow Y$ of E with an open subset $Y \subset Z$ and containing the zero section, such that

$$\begin{array}{ccc}
 E & \xrightarrow{\phi} & Y \\
 \pi \searrow & & \nearrow \pi \\
 & M &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 E & \xrightarrow{\phi} & Y \\
 \swarrow O(E) & & \searrow O(E) \\
 & M &
 \end{array}$$

commute. (i.e. ϕ is fibre preserving and maps $O(E)$ to $O(E)$ identically.)

Lemma 1 A singular bundle (π, E, M) over M is compressible.

Proof Let U be an open neighbourhood of $O(E)$. Choose a norm on E as in Lemma 1.4.2 and for each $x \in M$ choose U_x, ϕ, k_x, K_x , as in Lemma 1.4.3. Then if U_x is locally Euclidean, $\phi(U)$ is open in $U_x \times \mathbb{R}$ and contains $(x, 0)$. Hence

there exists $U'_x \subset U_x$ and an open ball $B(\varepsilon(x))$ (centre 0, radius $\varepsilon(x)$) $\subset \mathbb{R}$ so that $(x, 0) \in U'_x \times B(\varepsilon(x)) \subset \varphi(U)$. Similarly if U_x contains a singular point then $\varphi(U)$ is open in $A(U_x)$ and contains $(x, 0)$. So there exists $U'_x \subset U_x$ and a number $\varepsilon(x) > 0$ so that $(x, 0) \in A_{\varepsilon(x)}(U'_x) \subset \varphi(U)$.

If $v \in \pi^{-1}(U'_x)$ and $\|v\| < k_x \varepsilon(x)$, then $\|\varphi(v)\| < \varepsilon(x)$ so $\varphi(v) \in \varphi(U)$ and therefore $v \in U$. Let $D_y(\varepsilon) = \pi^{-1}(y) \cap \|\cdot\|^{-1}[0, \varepsilon)$ for all $y \in U_x$ (where $\varepsilon = \varepsilon(x)$), and let $\varepsilon'(x) = k_x \varepsilon(x)$. Then $D_y(\varepsilon'(x)) \subset U$. Let $\{\theta'_i\}$ be a partition of unity subordinate to $\{U'_x\}$. Choose U'_{x_i} containing the support of θ'_i for each i . Then

$$\eta(x) = \sum_i \theta'_i(x) \varepsilon'(x_i)$$

is a smooth function and for all $x \in M$

$$\eta(x) \leq \max_{x \in U_{x_i}} \varepsilon'(x_i) \quad \text{so} \quad D_x(\eta(x)) \subset U.$$

Let $\Phi : [0, \infty) \longrightarrow [0, 1)$ be a smooth function which is the identity on a neighbourhood of 0.

$$\text{Then } g : v_x \longmapsto \eta(x) \cdot \frac{\Phi(\|v_x\|)}{\|v_x\|} \cdot v_x$$

(multiplication of v_x by a scalar using the linear structure on each fibre) is a compression, and if $v_x \in \pi^{-1}(x)$ then $\|g(v_x)\| \leq \eta(x)$, so $g(v_x) \in D_x(\eta(x)) \subset U$. ||

If M^{n-1} is a manifold w.s. and N^n is a manifold w.b., and $H : M \times \mathbb{R} \longrightarrow N$ a smooth map, then we say that H is an isotopy (of embeddings) between H_0 and H_1 if

a) for each t , $H_t : M \longrightarrow N : x \rightsquigarrow H(x,t)$ is an embedding,

b) there exist t_0, t_1 , $0 \leq t_0 < t_1 \leq 1$, such that $H_t = H_0$ if $t \leq t_0$, and $H_t = H_1$ if $t \geq t_1$.

We say two embeddings $f : M \longrightarrow N$ and $g : M \longrightarrow N$ are isotopic ($f \approx g$) if there exists an isotopy H as above such that $H_0 = f$ and $H_1 = g$.

Let N^n be a manifold w.b. and M^{n-1} a submanifold w.s.. Let (π, E, M) be a singular bundle, and Z an open neighbourhood of the zero section. An isotopy $H_t : Z \longrightarrow N$ of open embeddings such that each H_t is a tubular neighbourhood map for M will be called an isotopy of tubular neighbourhoods.

Let $t \neq 0 \in \mathbb{R}$, and (π, E, M) be a singular bundle over M . Then the map

$$t : E \longrightarrow E : (x, v) \rightsquigarrow (x, tv)$$

is a singular bundle isomorphism.

Proposition 2 Let M be a manifold w.s. $(S(M)$ consists of isolated points) and (π, E, M) , (π', E', M) singular bundles over M . Let $f : E \longrightarrow E'$ be a tubular neighbourhood of M in E' .

Then there exists an isotopy of tubular neighbourhoods $H_t : E \longrightarrow E'$ such that $H_1 = f$ and H_0 is a singular bundle isomorphism.

Proof We define H by $H_t = t^{-1} \circ f \circ t$, for $t \neq 0$. Then, when it is defined, H_t is a tubular neighbourhood map since t and t^{-1} are singular bundle isomorphisms (on E and E' resp.) and $f : E \longrightarrow E'$ is a tubular neighbourhood. We investigate what happens near $t = 0$.

Given $x \in M$, suppose that we can find a Euclidean neighbourhood U' of x over which E' admits a trivialisation $U' \times \mathbb{R}$. We can then find a smaller Euclidean neighbourhood U of x and an open ball B about 0 in the fibre of E such that E admits a trivialisation $U \times \mathbb{R}$ over U and such that the representation f' of f on $U \times \mathbb{R}$ maps $U \times B$ into $U' \times \mathbb{R}$. This is possible because of the continuity of f . On $U \times B$ we can represent f' by $f'(x, v) = (\phi(x, v), \psi(x, v))$ with $\phi(x, 0) = x$ and $\psi(x, 0) = 0$ for all $x \in M$. Note

that for all $(x, v) \in U \times \mathbb{R}$ there exists t_0 so that $t(x, v) = (x, tv) \in U \times B$ for all $t < t_0$.

We can represent H_t locally on $U \times B$ as $H_t'(x, v) = (\phi(x, tv), t^{-1} \circ \psi(x, tv))$. ϕ is then a smooth map in the variables x, t, v , even at $t = 0$. The second component can be written

$$\begin{aligned} t^{-1}\psi(x, tv) &= t^{-1}(\psi(x, tv) - \psi(x, 0)) \\ &= t^{-1} \int_0^1 D_2\psi(x, stv) \cdot tv \, ds \\ &= \int_0^1 D_2\psi(x, stv) \cdot v \, ds \end{aligned}$$

But this is also a smooth map, even at $t = 0$. Thus we can define $H_0'(x, v) = (x, D_2\psi(x, 0) \cdot v)$. Since f is an embedding, $D_2\psi(x, 0) : \mathbb{R} \longrightarrow \mathbb{R}$ is a linear isomorphism. Hence (H_0', id) is a singular bundle isomorphism on its domain of definition.

Now suppose we have a neighbourhood $U' \subset C(n, \lambda)$ of x over which E' has trivialisation $A(U')$. Then we can find a smaller open neighbourhood $U \subset C(n, \lambda)$ of x and an $\varepsilon > 0$ such that E admits a trivialisation $A(U)$ over U and the representative f' of f on $A(U)$ maps $A_\varepsilon(U)$ into $A(U')$. Again we use continuity of f . On $A_\varepsilon(U)$ we can represent f' by

$f'(x,v) = (\phi(x,v), \psi(x,v))$ where $\phi(x,v) \in M = O(E')$, and $\psi(x,v) \in \pi'^{-1}(\phi(x,v))$. As in the Euclidean case we have $\phi(x,0) = x$ and $\psi(x,0) = 0$. Also for all $(x,v) \in A(U)$ we can find t_0 so that $t(x,v) = (x,tv) \in A_\varepsilon(U)$ for all $t < t_0$.

We can represent H_t locally on $A_\varepsilon(U)$ by $H'_t(x,v) = (\phi(x,tv), t^{-1} \circ \psi(x,tv))$. Again ϕ is a smooth map in x, t, v , even at $t = 0$. The fibre coordinate can be written

$$\begin{aligned} t^{-1}\psi(x,tv) &= t^{-1}(\psi(x,tv) - \psi(x,0)) \\ &= t^{-1} \int_0^1 D_v \psi(x,stv) \cdot tv \, ds \\ &= \int_0^1 D_v \psi(x,stv) \cdot v \, ds \quad , \quad v \neq 0 \quad , \end{aligned}$$

where $D_v \psi(x,u)$ is the directional derivative (in the linear structure on the fibre containing v) of ψ along the direction v in the fibre containing v . Also we define $t^{-1}\psi(x,0) \equiv 0$. Then $t^{-1}\psi(x,tv)$ is again a smooth map, even at $t = 0$. Thus we can define

$$\begin{aligned} H'_0(x,v) &= (x, D_v \psi(x,0) \cdot v) \quad , \quad v \neq 0, \\ H'_0(x,0) &= (x,0) \quad . \end{aligned}$$

Since f is an embedding, $D_v \psi(x,0) : \mathbb{R} \longrightarrow \mathbb{R}$ or $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a linear isomorphism.

Hence (H_0, id) is a singular bundle isomorphism

and the result follows by using a smooth function $\sigma(t)$ with $\sigma(t) = 0$ for $t \leq 0$, $\sigma(t) = 1$ for $t \geq 1$, to get the isotopy in standard form. ||

Theorem 3 Let M^{n-1} be a submanifold w.s. of the manifold w.b. N^n . Let (π, E, M) and (π', E', M) be two singular bundles over M , and $f : E \longrightarrow N$, $g : E' \longrightarrow N$ two tubular neighbourhoods of M in N . Then there exists an isotopy of tubular neighbourhoods $H_t : E \longrightarrow N$ and a singular bundle isomorphism $\lambda : E \longrightarrow E'$ such that $H_1 = f$ and $H_0 = g \circ \lambda$.

Proof Observe that $f(E)$ and $g(E')$ are open neighbourhoods of M in N . Let $U = f^{-1}(f(E) \cap g(E'))$ and let $\phi : E \longrightarrow U$ be a compression. Let ψ be the composite map $f \circ \phi : E \longrightarrow N$. Then ψ is a tubular neighbourhood of the type considered in Proposition 1.6.2. Thus there exists an isotopy of tubular neighbourhoods of M , $G_t : E \longrightarrow E'$ such that $G_1 = g^{-1} \circ \psi$ and G_0 is a singular bundle isomorphism. Now $F_t = g \circ G_t : E \longrightarrow N$ is an isotopy of tubular neighbourhoods with $F_1 = \psi$, $F_0 = g \circ G_0$.

Thus ψ and $g \circ G_0$ are isotopic by an

isotopy of tubular neighbourhoods. Similarly ψ and $f \circ J_0$ are isotopic, for some singular bundle isomorphism $J_0 : E \longrightarrow E'$. Consequently we can find an isotopy of tubular neighbourhoods from $g \circ G_0$ to $f \circ J_0$, K_t say. Then $K_t \circ J_0^{-1} = H_t$, gives the desired isotopy from $g \circ G_0 \circ J_0^{-1}$ to f , and $\lambda = G_0 \circ J_0^{-1} : E \longrightarrow E'$ is a singular bundle isomorphism. ||

1.7 Trivial Singular Bundles

We say that a singular bundle (π, E, M) over M is trivial if we can find a cover $\{(U_i, h_i)\}$ of M by coordinate charts, and trivialisations $\{\phi_i\}$ w.r.t. which all the transit functions ϕ_{ij} are mappings onto the identity : $\mathbb{R} \longrightarrow \mathbb{R}$.

Trivial singular bundles will be of great importance later. In this section we will investigate the existence of certain functions on singular bundles, and then the conditions for a manifold w.s. to admit a trivial singular bundle.

Theorem 1 If (π, E, M) is a trivial singular bundle, then there exists a smooth function $H : E \longrightarrow \mathbb{R}$ such that $H^{-1}(0) = M \subset E$ and $\partial H_- = M = \partial H_+$, where

$$H_+(H_-) = \{x \in E : H(x) \geq 0 \text{ (} H(x) \leq 0 \text{)}\}$$

and ∂ denotes the topological boundary of sub-

sets of E .

Proof Choose $\{(U_i, h_i)\}$ and $\{\phi_i\}$ w.r.t. which all the ϕ_{ij} are mappings onto $\text{id}_{\mathbb{R}}$.

If U_i is locally Euclidean define H on $\pi^{-1}(U_i)$ by $H(x) = \pi_2 \circ \phi_i(x)$. If U_i contains a singular point define H on $\pi^{-1}(U_i)$ by $H(x) = q \circ \phi_i(x)$ (see section 1.3) . Since we may assume that each singular point is contained in just one coordinate neighbourhood, it suffices to show that H is well defined on intersections of coordinate neighbourhoods which do not contain a singular point. But for $x \in U_i \cap U_j$ we have $\phi_j(x) = \phi_{ji}(x) \circ \phi_i(x) = \phi_i(x)$. Hence H is well defined and is clearly a smooth function satisfying the conditions. ||

Theorem 2 If (π, E, M) is a singular bundle on which there exists a continuous function H such that $H^{-1}(0) = M \subset E$ and $\partial H_- = M = \partial H_+$, then (π, E, M) is a trivial singular bundle.

Proof Choose a cover of M by coordinate neighbourhoods $\{(U_i, h_i)\}$ with trivialisations ψ_i for E . Choose a smooth norm on E and let

$T = \parallel \parallel^{-1}[0,1]$. Then T is a closed tubular neighbourhood of M in E . Thus T is a manifold w.b. and ∂T can be written

$$\partial T = \partial T_+ \cup \partial T_- ,$$

where $\partial T_+ = \{x \in \partial T : H(x) > 0\}$ and $\partial T_- = \{x \in \partial T : H(x) < 0\}$. $\partial T_+ \cap \partial T_- = \emptyset$ so ∂T is the disjoint union of these two components, neither of which is empty.

We define a new set of trivialisations φ_i for E as follows: If U_i is locally Euclidean define φ_i to be a map $\pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}$ such that $\varphi_i \circ \psi_i^{-1}(x,t) = (x, \varphi_{ix} \circ \psi_{ix}^{-1}(t))$ where $\varphi_{ix} \circ \psi_{ix}^{-1} \in L(\mathbb{R}, \mathbb{R})$ and $\varphi_{ix}^{-1}(1) \in \partial T_+$ for all $x \in U_i$. If U_i contains a singular point define φ_i to be a map $\pi^{-1}(U_i) \longrightarrow A(U_i)$ such that

a) if x is a Euclidean point then

$$\varphi_i \circ \psi_i^{-1}(x,t) = (x, \varphi_{ix} \circ \psi_{ix}^{-1}(t)) \text{ where } \varphi_{ix} \circ \psi_{ix}^{-1} \in L(\mathbb{R}, \mathbb{R}) \text{ and } \varphi_{ix}^{-1}(1) \in \partial T_+ ,$$

b) if x is a singular point then

$$\varphi_i \circ \psi_i^{-1}|_{p^{-1}(x)} : D(n-1, \lambda) \longrightarrow D(n-1, \lambda) \text{ is fibre linear and maps } (x,t) \rightsquigarrow (x,t') , \text{ where } (x,t) \in S^{\lambda-1} \times \mathbb{R}_+ / S^{\lambda-1} \times \{0\} = D^\lambda \subset D(n-1, \lambda) \text{ and } \varphi_i^{-1}(x,1) \in \partial T_+ ;$$

$$\text{and } \varphi_i \circ \psi_i^{-1}|_{p^{-1}(x)} \text{ maps } (y,s) \rightsquigarrow (y,s') , \text{ where}$$

$$(y, s) \in S^{\mu-1} \times \mathbb{R} / S^{\mu-1} \times \{0\} = D^{\mu} \subset D(n-1, \lambda)$$

$$\text{and } \varphi_i^{-1}(y, -1) \in \partial T_- .$$

We assert that the φ_i so defined are indeed a set of trivialisations for E and that for each $x \in E$, $\varphi_{ij}(x) = \text{id}_{\mathbb{R}}$.

To prove that the φ_i are trivialisations it is sufficient to prove parts 1,b and 2,b,d of the definition (section 1.4). 1,b and 2,b follow from the fact that ∂T_+ is a smooth submanifold of E . For 2,d we need to observe that at each Euclidean point $x \in U_i$, $\varphi_i^{-1}(x, 1) \in \partial T_+$ implies that $\varphi_i^{-1}(x, -1) \in \partial T_-$. But certainly $\varphi_i^{-1}(x, -1) \in \partial T$ since its norm is 1, and it cannot be in ∂T_+ (to see this look at the sign of H on $\pi^{-1}(x)$). Since H changes from positive to negative at $(x, 0)$ only, H must be negative at $\varphi_i^{-1}(x, -1)$. Then 2,d follows from the fact that ∂T_+ and ∂T_- are smooth submanifolds of E .

Now by construction $\varphi_{ij}(x) = \pm 1$, and also at each Euclidean point $\varphi_{ij}(x) = 1$. Hence E is a trivial singular bundle. ||

Theorem 3 Suppose that M is a compact manifold w.s. and (π, E, M) is a singular bundle. Suppose

that on E we have a continuous function H such that $H^{-1}(0) = M$, $\partial H_- = M = \partial H_+$, and H is smooth except on M and has no critical points in $E - M$. Then there exists a tubular neighbourhood U of M in E and a smooth function H' on E such that $H'^{-1}(0) = M$, $\partial H'_- = M = \partial H'_+$, $H' = H$ outside U , and H' has no critical points in $E - M$.

Remark For the definition of a critical point of a real valued function see section 2.1. We do not use here any of the results of later sections.

Proof By the continuity of H there exists $\varepsilon > 0$ such that $H^{-1}[-\varepsilon, \varepsilon]$ is a closed tubular neighbourhood of M , and the submanifolds $H^{-1}(e)$, $|e| < \varepsilon$, are transverse to the fibres of E . Putting $U = H^{-1}[-\varepsilon, \varepsilon]$, ∂U is a smooth submanifold of E . Remove U by cutting along ∂U .

Now since H is continuous on E , E has a trivial bundle structure constructed as in Theorem 1.7.2. Hence we can define a smooth function K on E as in Theorem 1.7.1, and $V = K^{-1}[-\varepsilon, \varepsilon]$ is a closed tubular neighbourhood of M in E . It follows from Proposition 1.6.2 that U and V

are diffeomorphic. The restriction of this diffeomorphism to the boundaries gives a diffeomorphism $h : \partial U \longrightarrow \partial V$, and by examining the constructions involved we see that diffeomorphism and its restriction can both be taken to be fibre preserving. Also we can choose h so that $\partial U_+ \longrightarrow \partial V_+$ and $\partial U_- \longrightarrow \partial V_-$.

We now glue V to $E - U$ by h so that the function H' defined by

$$H'(x) = K(x) \quad \text{on } V,$$

$$H'(x) = H(x) \quad \text{on } E - U,$$

is smooth on the resulting manifold $E' \cong E$.

H' is the desired function. ||

We now look at existence and uniqueness problems for trivial singular bundles.

Proposition 4 If M is a manifold w.s., then there exists a trivial singular bundle over M .

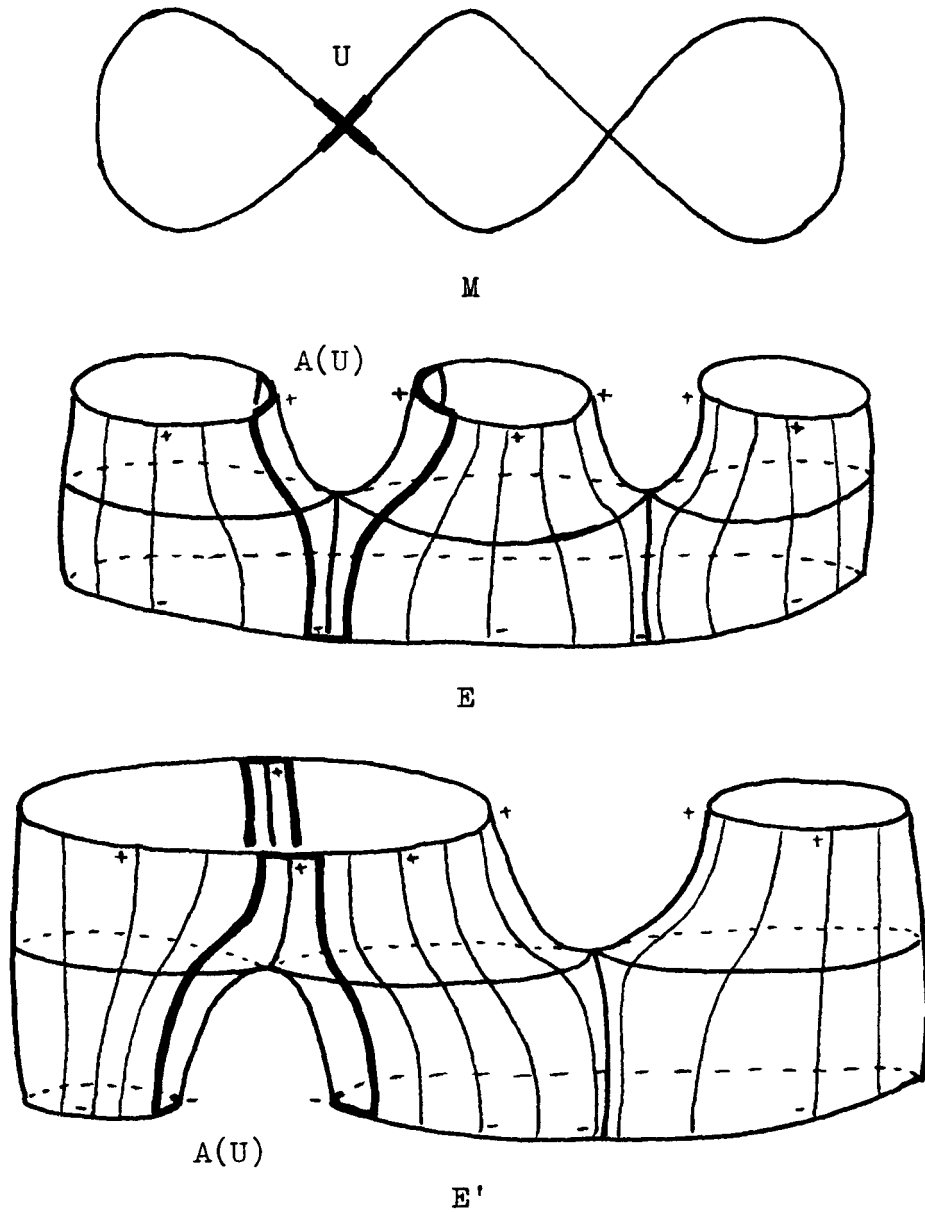
Proof Take a coordinate neighbourhood U_i about each singular point y_i . For each i , choose $\varepsilon_i > 0$ so that $V_i = \{x \in C(n-1, \lambda) : |x| = \varepsilon_i\}$ is contained in U_i . Cut M along V_i for each i . The result is a manifold w.b. M' and

a collection of neighbourhoods U'_i of singular points. We form the manifold w.b. $M' \times \mathbb{R}$ and glue in the neighbourhoods $A(U'_i)$ in such a way that the result is always a trivial singular bundle.

Let V be a connected boundary component of $A(U'_i)$. Then V is a product $W \times \mathbb{R}$ as is the boundary component of $M' \times \mathbb{R}$ to which it is to be glued. Hence we can glue V to the corresponding boundary of $M' \times \mathbb{R}$ so that the base manifold w.s. is rejoined as before and corresponding fibres are glued by the identity map. The result of such glueings is clearly a trivial singular bundle. ||

Proposition 5 If M has $r > 0$ singular points, then there are $< 4^r$ distinct trivial singular bundles over M .

Proof We use the notation of the last proposition. Suppose that the set U'_i is a neighbourhood of a singular point y_i . If the removal of $A(U'_i)$ disconnects E , we can form a new singular bundle by changing the direction of every fibre on one component and glueing $A(U'_i)$ into the new space so as to form a trivial singular bundle. (See fig.1.)



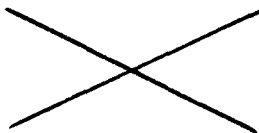
E and E' are trivial singular bundles over M . E' is obtained from E by cutting the associated neighbourhood $A(U)$ from E along the fibres shown and glueing it back after reversing directions on the fibres of $A(U)$.

Figure 1

Note that the new trivial singular bundle may or may not be diffeomorphic with the original one.

If M has dimension ≥ 2 then each neighbourhood of a singular point has at most two boundary components. Hence there are at most four distinct trivial singular bundles due to the choice of glueing of $A(U'_i)$. If M has r singular points, it follows that there are at most 4^r distinct trivial bundles over M .

If M has dimension 1 then every U'_i is diffeomorphic to



Let us remove an $A(U'_i)$ and consider the number of ways we can glue it back in to E to get a trivial singular bundle. If removal of $A(U'_i)$ does not disconnect E then there are two possible ways of glueing $A(U'_i)$ into E to get a trivial singular bundle. (In fact both give the same result) If removal of $A(U'_i)$ disconnects E then we have four possibilities, for the boundaries of $E - A(U'_i)$ to which $A(U'_i)$ is to be glued occur in pairs and on each pair the glueing must be consistent. It follows as before that M has at most 4^r trivial singular bundles. ||

Proposition 6 If M is a manifold w.s. with one singular point, then there exists a unique trivial singular bundle over M .

Proof We use the notation of the previous two propositions. Let U' be the neighbourhood of the singular point y . Then $M - U'$ is a manifold with one or two connected components, and $E - A(U')$ is a product. Hence the trivial singular bundle is the same however we glue in $A(U')$. ||

We define a manifold w.s. M to be orientable if it has a trivial singular bundle (π, E, M) such that E is an orientable manifold.

Proposition 7 If M is an orientable manifold w.s. and (π, E, M) is a trivial singular bundle over M , then E is an orientable manifold.

Proof We show that the orientability of E is independent of the way in which the $A(U'_i)$ are glued into E .

If E is orientable then for every closed path P in E we can cover P by coordinate neighbourhoods V_j such that on the intersection

of any two V_j their Jacobian is positive, and if P passes through any $A(U_i')$ then we choose that $A(U_i')$ with appropriate orientation as one of the V_j . By homotopy of P if necessary, we may suppose that whenever P enters a $A(U_i')$ it leaves by a different boundary component.

Now suppose that removal of $A(U_i')$ does not disconnect E . Then there are two possible ways of glueing $A(U_i')$ into E . If P passes through $A(U_i')$ then the Jacobian as P enters is positive or negative according to the glueing chosen, but the Jacobian as P exits is resp. positive or negative also. Hence the glueing does not affect the orientability of E .

Suppose that $A(U_i')$ disconnects E and that $\dim M > 1$. Then $A(U_i')$ has two boundary components and P always traverses $A(U_i')$ by entering at one boundary and exiting at the other. Now the result of such a traverse is either 0, 1, or 2 negative Jacobians. But since P is a closed path P must traverse $A(U_i')$ an even number of times, so the path P will always be orientable. If $\dim M = 1$ then another possibility occurs, viz. that P enters from component C of $E - A(U_i')$ by one boundary component and returns to

C by another boundary component. In this case the result is either 0 or 2 negative Jacobians and again P is an orientable path.

The result follows. ||

Let M^n be a manifold w.s. and (π, E, M) a trivial singular bundle over M . Let $\| \cdot \|$ be a smooth norm on E and let $T = \| \cdot \|^{-1}[0, 1]$. T is a smooth submanifold w.b. of E of dimension $n+1$. Let N be a connected component of ∂T (by Theorem 1.7.1 there are at least two such). We call T the canonical neighbourhood of M in E , and N a surface of T (or, by abuse of language, a surface of M). A function on E of the type discussed above induces a decomposition of ∂T as $\partial T_+ \cup \partial T_-$. In this case we say that a surface N is positive or negative according as $N \subset \partial T_+$ or $N \subset \partial T_-$.

Chapter 2

Hamiltonian Systems and Graphs

In the remainder of this paper M will denote a connected manifold w.b., unless stated to be otherwise. We will also assume that ∂M is compact.

In this chapter we leave manifolds w.s. and review the definitions and results which we will need in the next chapter. We begin by looking at real valued functions on manifolds w.b. For further information see [3]. Next we give a brief survey of Hamiltonian Dynamical Systems, and of topologies on the space of functions on a manifold w.b. and on the space of vector fields on a manifold w.b. For further information see [6]. In general our notation will follow that of Abraham and Marsden in [6]. Finally we give some results on graphs.

2.1 Functions on Manifolds w.b.

Let $\mathcal{F}^r(M)$, $r \geq 2$, denote the set of C^r functions $f : M \longrightarrow \mathbb{R}$, TM the tangent bundle

of M , and T^*M the cotangent bundle. Let $\mathcal{X}^r(M)$ denote the set of C^r vector fields on M and $\mathcal{X}_r^*(M)$ the C^{r-1} -forms on M , $r \geq 1$.

If $f \in \mathbb{F}^{r+1}(M)$, $r \geq 1$, then $df \in \mathcal{X}_r^*(M)$, and we say that $m \in M$ is a critical point of f if $df(m) = 0$. If (x^1, \dots, x^n) are coordinates on a neighbourhood of m , we have

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n$$

and so m is a critical point of f if and only if $\frac{\partial f}{\partial x^1} = \dots = \frac{\partial f}{\partial x^n} = 0$.

$e \in \mathbb{R}$ is called a critical value of f if there exists $m \in f^{-1}(e)$, m a critical point. $e \in \mathbb{R}$ is called a boundary value of f if there exists $m \in \partial M$ with $f(m) = e$. $e \in \mathbb{R}$ is called a regular value of f if it is neither a critical value nor a boundary value.

We say that a C^r , $r \geq 2$, function $H : M \longrightarrow \mathbb{R}$ is a Hamiltonian function if H is locally constant on ∂M and there exists an open neighbourhood U of ∂M which contains no critical points. Let $\mathcal{H}^r(M)$ denote the set of C^r Hamiltonian functions on M . Clearly $\mathcal{H}^r(M) \subset \mathbb{F}^r(M)$ and if $\partial M = \emptyset$ then $\mathcal{H}^r(M) = \mathbb{F}^r(M)$. In the

next section we define a Hamiltonian vector field on a manifold w.b. to be a vector field induced by a Hamiltonian function. The advantage of restricting ourselves to vectorfields generated by Hamiltonian functions is that their flows are well behaved at the boundary, i.e. an orbit is either contained in the boundary or disjoint from it. In particular the flow on the boundary is complete.

Let $H \in \mathcal{H}^r(M)$ and let C be the set of critical points of H . Then $C \subset \overset{\circ}{M}$, C is closed in M , and so $M' = M - C$ is a manifold w.b. and a submanifold w.b. of M . $\partial M' = \partial M$.

An energy surface E_e of H in M is a connected component of $H^{-1}(e)$ (in M) intersected with M' . Thus, in general, an energy surface is not itself connected. A regular energy surface Σ_e of H in M is a connected component of $H^{-1}(e)$ in M such that $\Sigma_e \cap (C \cup \partial M) = \emptyset$. Hence a regular energy surface is an energy surface. A singular energy surface S_e of H in M is an energy surface which is not regular. An energy manifold is a connected component of an energy surface. Thus a regular energy surface is an

energy manifold, and a singular energy surface is, in general, a disjoint union of energy manifolds.

Energy surfaces have the following elementary properties:

Proposition 1 If $H \in \mathcal{H}^r(M)$, $r \geq 2$, then every point $m \in M$ is either

- a) a critical point of H , or
- b) on a regular energy surface of H , or
- c) on a singular energy surface of H . ||

Proposition 2 Let $H \in \mathcal{H}^r(M)$, $r \geq 2$, and let S_e be a singular energy surface of H and Q a connected component of ∂M . If $S_e \cap Q \neq \emptyset$, then $S_e = Q$.

Proof Since H is locally constant on ∂M , if $S_e \cap Q \neq \emptyset$ then $Q \subset S_e$. Also, about each point $x \in Q$ we can find coordinates (x^1, \dots, x^n) , $x^1 \geq 0$, and $x = 0$. At each $\text{boundary point } y \in U_x$ - the coordinate neighbourhood - we have $\frac{\partial H}{\partial x^1}(y) = 0$ for $i \neq 1$.

But y is not a critical point of H , so $\frac{\partial H}{\partial x^1}(y)$

$\neq 0$. Hence there exists an open neighbourhood

W_x of $Q \cap U_x$ such that $H^{-1}(e) \cap W_x = Q \cap W_x$.
Hence there exists an open neighbourhood W of
 Q such that $H^{-1}(e) \cap W = Q$. So $S_e \subset Q$. ||

Proposition 3 a) An energy surface E_e of $H \in \mathcal{H}^r(M)$ is an orientable submanifold of M of codimension one, and a topologically closed submanifold of M' .

b) If E_e is a regular energy surface, or a singular energy surface with $E_e \cap \partial M \neq \emptyset$, then E_e is a connected topologically closed submanifold of M .

c) For any energy surface E_e there exists a neighbourhood U of E_e such that $H^{-1}(e) \cap U = \overline{E_e}$.

Proof Use Proposition 2.1.2 and the results in Abraham and Marsden [6,p.111]. Note that the results in Abraham and Marsden apply to energy surfaces generally. ||

The closure (in M) of an energy surface will be called a closed energy surface. Thus a closed energy surface will be ~~either~~ a manifold if it does not contain any critical points.
~~or a manifold w.s.~~ If $m \in C$ we define the

closed energy surface containing m to be the set $\{m\} \cup E_e : m \in \bar{E}_e$. Hence if there does not exist an \bar{E}_e containing m then the closed energy surface containing m is $\{m\}$. In this case we say that m is an isolated critical (or singular) point.

2.2 Non-degeneracy and the Morse Lemma

A critical point $m \in M$ of $H \in \mathbb{H}^r(M)$, $r \geq 2$, is said to be non-degenerate if there exists a local coordinate chart with coordinates (x^1, \dots, x^n) in a neighbourhood of m such that at m the matrix

$$\left(\frac{\partial^2 H}{\partial x^i \partial x^j}(m) \right)$$

is non-singular. This definition is independent of the choice of coordinates.

Morse Lemma 2.2.1 [3] If $m \in M$ is a non-degenerate critical point of $H \in \mathbb{H}^r(M)$, $r \geq 2$, then there exists a local coordinate system (x_1, \dots, x_n) of C^r coordinates in a neighbourhood of m w.r.t. which H has the form

$$H(x_1, \dots, x_n) = H(0) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2$$

where $0 \leq \lambda \leq n$ and $m = 0$.

||

λ is called the Morse index of m .

If $H \in \mathcal{H}^r(M)$ is a function all of whose critical points are non-degenerate then we call H a non-degenerate Hamiltonian function. Let $\mathcal{H}_N^r(M)$ denote the set of all non-degenerate Hamiltonian functions.

Proposition 2 Let $H \in \mathcal{H}_N^r(M)$, $r \geq 2$, and let S_e be a singular energy surface of H with $S_e \cap \partial M = \emptyset$. Then \bar{S}_e is a submanifold w.s. of M .

Proof Propositions 2.1.3 and 2.2.1 .

||

If $m \in \bar{S}_e$ is a critical point of H with Morse index λ , and $\mu = \min(\lambda, n-\lambda)$, then μ is the singularity index of m in \bar{S}_e .

2.3 Gradient and Inverse Gradient

Let M be a manifold w.b., g a Riemannian metric on M , and $H \in \mathcal{H}^r(M)$, $r \geq 2$. g and H induce on M a gradient vector field, $\text{grad } H$, whose orbits are always perpendicular (w.r.t. g) to the energy surfaces of H . The set of critical points of $\text{grad } H$ is identical to the set of

critical points of H . Thus on $M' = M - C$ we can define the inverse gradient vector field, $\text{grin } H$, by $\text{grin } H = \frac{\text{grad } H}{\|\text{grad } H\|^2}$. Thus the flow φ for $\text{grin } H$ has the same orbits as the flow for $\text{grad } H$, but a different time parametrization. Along these orbits we have

$$\begin{aligned} \frac{dH(m)}{dt} &= \frac{dH(\varphi_t(m))}{dt} = (\text{grin } H)(H) \\ &= g(\text{grin } H, \text{grad } H) = 1. \end{aligned}$$

This is independent of m , and so φ_t maps $H^{-1}(e)$ onto $H^{-1}(e+t)$ whenever $\varphi_t|_{H^{-1}(e)}$ is defined. Note that $\text{grad } H$ and $\text{grin } H$ do not in general have complete flows.

Proposition 1 Let M be a manifold w.b., g a Riemannian metric on M , and $H \in \mathcal{H}^r(M)$, $r \geq 2$. Let φ be the flow for $\text{grin } H$. If Σ_e is a regular energy surface of H and $\varphi_t(m)$ is defined for all $m \in \Sigma_e$, then $\varphi_t|_{\Sigma_e}$ is a diffeomorphism of Σ_e onto Σ_{e+t} , where Σ_{e+t} is either a regular energy surface or a singular energy surface contained in ∂M .

Proof Consider the orbit of $\text{grin } H$ through $m \in \Sigma_e$. Suppose $t > 0$. Then if an orbit terminates, i.e.

if $\phi_t(m)$ is not defined for some t , then it must terminate at a critical point or on a boundary component. Suppose that there does not exist a critical point in the component of $H^{-1}[e, e+t]$ containing Σ_e , and that ∂M intersects this set in at most Σ_e and E_{e+t} . Then each orbit through Σ_e also passes through E_{e+t} . Thus $\phi_t|_{\Sigma_e}$ is clearly a diffeomorphism. ||

Note that if there exists a critical point between Σ_e and E_{e+t} then there is an orbit which terminates at the critical point and $\phi_t|_{\Sigma_e}$ is not well defined.

2.4 Topologies on $H^r(M)$

We consider three cases in which we wish to put a topology on $H^r(M)$, $r \geq 2$.

Suppose that M is a closed manifold (i.e. compact, $\partial M = \emptyset$). We can choose a cover of M by a finite number of coordinate charts $\{U_i\}$. For each $H \in H^r(M)$ we define a C^s , $1 \leq s \leq r$, norm for H on each U_i by

$$\|H\|_i = \sup\{\|D^k H(x)\| : x \in U_i, 0 \leq k \leq s\},$$
and $\|H\| = \max_i\{\|H\|_i\}$ is then a global norm.

With this norm $\mathbb{H}^r(M)$ is a Banach space. The topology defined by the norm is called the (Whitney) C^s topology on $\mathbb{H}^r(M)$ and is independent of the charts used to define the norm.

$\mathbb{H}^r(M)$ with the C^s topology has the Baire property, viz. a residual set (a countable intersection of open-dense sets) in $\mathbb{H}^r(M)$ is dense. (For a different and more general approach see [6].)

Suppose that M is a compact manifold w.b. By the above argument we show that $\mathbb{F}^r(M)$ with the C^s norm is a Banach space. Let $G^r(M) = \{f \in \mathbb{F}^r(M) : f \text{ is locally constant on } \partial M\}$. Then $G^r(M)$ is a closed subspace of $\mathbb{F}^r(M)$ and hence a Banach space under the induced C^s norm. Now with the topology on $G^r(M)$ induced by this norm, $\mathbb{H}^r(M)$ is an open subset of $G^r(M)$ with the C^s topology when $s \geq 1$. For then $H \in \mathbb{H}^r(M)$ has no critical points in a neighbourhood U of ∂M . So if $H' \in \mathbb{H}^r(M)$ is sufficiently close to H in the C^s topology, $s \geq 1$, then, using elementary techniques from [7], there exists a neighbourhood U' of ∂M in which H' has no critical points.

~~non degenerate and keeping the function locally constant on ∂M . So again the result is a corollary of [7, p. 46].~~

$\mathcal{H}^r(M)$ with the induced topology has the Baire property. The induced topology on $\mathcal{H}^r(M)$ is again called the (Whitney) C^S topology.

Suppose that M is a (non-compact) manifold. Choose a sequence $K_0 \subset K_1 \subset \dots \subset K_i \subset \dots \subset M$ of compact sets with $K_0 = \emptyset$, $\bigcup_i K_i = M$, $\bar{K}_i \subset \overset{\circ}{K}_{i+1}$, and $\overset{\circ}{K}_i \neq \emptyset$ if $i \neq 0$. If $H \in \mathcal{H}^r(M)$ and $\delta : M \rightarrow \mathbb{R}_+$ is a positive continuous function, define $\delta_i = \inf\{\delta(x) : x \in K_i - \overset{\circ}{K}_{i-1}\}$ and

$$N(H, \delta) = \bigcap_{i=1}^{\infty} \{K : \|H - K\|_i < \delta_i\}$$

where $\|\cdot\|_i$ is a C^S norm on the compact set $K_i - \overset{\circ}{K}_{i-1}$.

The sets of the form $N(H, \delta)$ define a basis for a topology on $\mathcal{H}^r(M)$, called the (Whitney) C^S topology. The C^S topology is independent of the decomposition of M and of the C^S norm chosen on each $K_i - \overset{\circ}{K}_{i-1}$. $\mathcal{H}^r(M)$ with the C^S topology has the Baire property. (For details see [10]. We are dealing with cross-sections of the trivial bundle $M \times \mathbb{R}$ rather than with vector fields, but the argument is identical.)

The theory below will be developed for two cases,

- a) M is a compact manifold w.b.,
- b) M is a non-compact manifold.

We hope that it will then be clear how to apply the theory in the most general case when M is a non-compact manifold w.b., ∂M ^{has} compact components.

2.5 Hamiltonian Dynamical Systems

In this section we give a brief survey of the theory of Hamiltonian dynamical systems. For proofs of the results mentioned below and for further information see Abraham and Marsden [6]. All the results quoted are given in [6] for manifolds but extend trivially to manifolds w.b.

A symplectic form on a manifold w.b. M is a non-degenerate closed two form ω on M . A symplectic manifold w.b. (M, ω) is a manifold w.b. M together with a symplectic form ω on M .

Let ω be a two form on M . Then if ω is non-degenerate M must be even dimensional with dimension $2n$ say, and then $\omega^n = \omega \wedge \omega \wedge \dots \wedge \omega$ is a volume on M . Any manifold w.b. possessing a volume is orientable. Hence a symplectic

manifold w.b. is even dimensional and orientable.

We will find it convenient to take

$$\Omega = \frac{(-1)^{n(n-1)/2}}{n!} \cdot \omega^n \quad \text{as the standard volume on } M.$$

Let ω be a two form on M . Define a map

$$\omega_b : TM \longrightarrow T^*M \quad \text{by} \quad \omega_b(m) : T_m M \longrightarrow T_m^*M, \quad \text{and}$$

$$[\omega_b(m) \cdot e](e') = \omega_m(e, e'), \quad \text{for } e, e' \in T_m M.$$

For $X \in \mathfrak{X}^r(M)$ define

$$X^b : M \longrightarrow T^*M \quad \text{by} \quad X^b(m) = 2 \omega_b(m) \cdot X(m).$$

$$\text{Define } b : \mathfrak{X}^r(M) \longrightarrow \mathfrak{X}_r^*(M) : X \rightsquigarrow X^b.$$

If ω is non-degenerate then ω_b is a vector bundle isomorphism, and we write $\omega_b^{-1} = \omega_{\#}$. b is an

$\mathbb{R}^r(M)$ -linear vector space isomorphism, and we write

$$b^{-1} = \# : a \rightsquigarrow a^{\#}.$$

Darboux Theorem Let ω be a non-degenerate two form on a $2n$ -manifold w.b. M . Then $d\omega = 0$ if and only if there exists a coordinate chart (U, h) in a neighbourhood of each $m \in M$ giving local coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ ($x^1 \geq 0$ if $m \in \partial M$) w.r.t. which $h(m) = 0$ and

$$\omega|_U = \sum_{i=1}^n dx^i \wedge dy^i. \quad ||$$

On a symplectic manifold w.b. the charts

whose existence is verified by Darboux Theorem are called symplectic charts, and the local coordinates are called local symplectic coordinates.

If (M, ω) and (N, ρ) are symplectic manifolds w.b., a smooth mapping $f : M \longrightarrow N$ is called symplectic if $f_* \rho = \omega$. A symplectic map is volume preserving and a local diffeomorphism.

Given any n -manifold w.b. M , T^*M has a natural symplectic structure. Let $\tau^* : T^*M \longrightarrow M$ be the natural projection. Suppose that $\alpha_m \in T_m^*M$, and $V_{\alpha_m} \in T_{\alpha_m}(T^*M)$. Define

$$\theta_{\alpha_m} : T_{\alpha_m}(T^*M) \longrightarrow \mathbb{R} : V_{\alpha_m} \xrightarrow{\quad} \alpha_m \cdot \tau^*(V_{\alpha_m})$$

$$\text{and } \theta : T^*M \longrightarrow T^*(T^*M) : \alpha_m \xrightarrow{\quad} \theta_{\alpha_m}.$$

Then $\theta \in \mathcal{X}_{\infty}^*(T^*M)$ and $\omega = -d\theta$ is a symplectic form on T^*M .

Let (M, ω) be a symplectic manifold w.b. and $X \in \mathcal{X}^r(M)$. We say that X is a Hamiltonian vector field if $X = (dH)^\#$ for some $H \in \mathcal{H}^{r+1}(M)$, $r \geq 1$, and we write $X = X_H$. Let $\mathcal{X}_H^r(M)$ denote the set of all C^r Hamiltonian vector fields. Clearly $\mathcal{X}_H^r(M) \subset \mathcal{X}^r(M)$. The notation $\mathcal{X}_H^r(M)$ will be taken to imply that (M, ω) is a symplectic manifold w.b. for some given or understood ω .

We say that a vector field X is locally Hamiltonian if for all $m \in M$ there exists a neighbourhood U of m and $H \in \mathcal{H}^{r+1}(M)$ such that $X|_U = X_H|_U$. Let $\mathcal{X}_{LH}^r(M)$ denote the set of all locally Hamiltonian vector fields which are C^r on M . Clearly $\mathcal{X}_H^r(M) \subset \mathcal{X}_{LH}^r(M)$.

If X is a locally Hamiltonian vector field, then

a) X^\flat is a closed one form.

b) For every flow box F for X , F_t is a symplectic diffeomorphism. Hence F_t is volume preserving and $\det F_t = 1$.

c) If H is a local Hamiltonian function for X in a neighbourhood U , then H is constant along integral curves for X in U . If $X = X_H$ is a Hamiltonian vector field then the energy surfaces of H are invariant manifolds of the flow.

If M is a compact manifold w.b. and $X \in \mathcal{X}_{LH}^r(M)$ then X is a complete vector field. In fact, we will suppose that all vector fields are complete unless we specifically withdraw this hypothesis.

Suppose that $X \in \mathcal{X}_{LH}^r(M)$ and that $X|_U = X_H|_U$ on some neighbourhood U . Let (V, ϕ) with

$V \subset U$ be a symplectic chart with local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$. Then a curve $c(t)$ on V is an integral curve for X if and only if

$$\begin{aligned}\frac{\partial q^i}{\partial t}(c(t)) &= \frac{\partial H(c(t))}{\partial p_i}, \\ \frac{\partial p_i}{\partial t}(c(t)) &= -\frac{\partial H(c(t))}{\partial q^i}, \quad i = 1, \dots, n.\end{aligned}$$

If M is a connected symplectic manifold and $X \in \mathcal{X}_H^r(M)$ then any two Hamiltonian functions H and H' inducing X differ by a constant.

2.6 Topologies on $\mathcal{X}_{LH}^r(M)$

We go about constructing a topology on $\mathcal{X}_{LH}^r(M)$ in the same way as we earlier constructed a topology on $\mathcal{H}^r(M)$. Again we wish to construct the topology for three cases, viz.

- a) M is a closed manifold.
- b) M is a compact manifold w.b.
- c) M is a non-compact manifold.

We give the details for the first case only, and leave it to the reader to construct the other two.

Let M be a closed manifold with symplectic form ω . We can choose a cover of M by a finite number of coordinate neighbourhoods U_i . For each

$X \in \mathcal{X}_{LH}^r(M)$ we can define a C^s ($0 \leq s \leq r$) norm for X on each U_i by

$$\|X\|_i = \sup\{\|D^k X(m)\| : m \in U_i, 0 \leq k \leq s\},$$

and then $\|X\| = \max_i \{\|X\|_i\}$ is a global norm on M .

With this norm $\mathcal{X}_{LH}^r(M)$ is a Banach space.

The topology defined by the norm is called the (Whitney) C^s topology on $\mathcal{X}_{LH}^r(M)$, and is independent of the charts used to define the norm. $\mathcal{X}_{LH}^r(M)$ with the C^s topology has the Baire property.

We give $\mathcal{X}_H^r(M) \subset \mathcal{X}_{LH}^r(M)$ the subspace topology.

Proposition 1 The map $\overline{X}^r : \mathcal{H}^{r+1}(M) \longrightarrow \mathcal{X}_H^r(M) : H \rightsquigarrow X_H$; where $r \geq 1$, $\mathcal{H}^{r+1}(M)$ has the C^{s+1} topology, and $\mathcal{X}_H^r(M)$ has the C^s topology, $0 \leq s \leq r$, is a continuous surjective map, and open when M is compact.

Proof Surjectivity is clear.

We prove continuity in the case when M is a non-compact manifold. With the above defined set of norms on M let

$$N(X_H, \delta) = \bigcap_i \{X_K : \|X_H - X_K\|_i < \delta_i\}.$$

Let $X_K \in N(X_H, \delta)$ and suppose that $\|X_H - X_K\|_i = \Delta_i < \delta_i$ for all i . Define $\varepsilon_i = \delta_i - \Delta_i > 0$ for each i , and suppose that $L \in N(K, \varepsilon)$. Then $\|L - K\|_i < \varepsilon_i$, for all i , implies that $\|X_L - X_K\|_i < \varepsilon_i$ for all i , implies that $\|X_L - X_H\|_i < \delta_i$ for all i . So $\overline{X}^r(N(K, \varepsilon)) \subset N(X_H, \delta)$. Hence the map is continuous.

When M is compact we can choose a finite cover of M by coordinate neighbourhoods (U_i, φ_i) , $i = 1, \dots, m$, and we may suppose that for each i $\varphi_i(U_i) = \{x \in \mathbb{R}^n : |x| < 1\}$. Use these coordinate neighbourhoods to define the topology on $\mathbb{H}^{r+1}(M)$ and $\mathbb{X}_H^r(M)$. Let $N(H, \delta) = \{K : \|H - K\| < \delta\}$ be a typical open set in $\mathbb{H}^r(M)$. Suppose that X_K is in its image and that $\|H - K\| = \Delta < \delta$. Let $\varepsilon = \delta - \Delta$. Choose ε' so that $2mn\varepsilon' < \varepsilon$, and suppose that $X_L \in N(X_K, \varepsilon')$. Choose the arbitrary constant for L so that $L(\varphi_1^{-1}(0)) = K(\varphi_1^{-1}(0))$. Then in U_1 $\|L - K\| < n\varepsilon'$.

Suppose w.l.o.g. that $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \dots, m-1$. Then in U_2 we have $\|L - K\| < 3n\varepsilon'$, etc. So on M we have $\|L - K\| < 2mn\varepsilon'$, so $L \in N(K, \varepsilon)$. Thus $N(X_K, \varepsilon') \subset \overline{X}^r(N(K, \varepsilon))$ and the map is open. ||

Thus we have a natural correspondence $H \longrightarrow X_H$, where (when M is connected) X_H determines H to within an arbitrary constant. Hence a property of H , such as non-degeneracy, can meaningfully be applied to X_H or to the flow it defines, and in future we will regard any property defined in terms of H (or X_H) as applying equally to the other.

Finally, we set $\mathcal{F}(M) = \mathcal{F}^\infty(M)$, $\mathcal{H}(M) = \mathcal{H}^\infty(M)$, etc. Then $\mathcal{H}(M)$ is dense in $\mathcal{H}^r(M)$, $r \geq 1$, and if U is an open (open-dense) set in $\mathcal{H}^r(M)$ then $V = U \cap \mathcal{H}(M)$ is open (open-dense) in $\mathcal{H}(M)$. Below we will only consider smooth functions and vector fields in $\mathcal{F}(M)$, $\mathcal{X}(M)$, etc. The results hold also for the non-smooth case when we make allowance for different degrees of differentiability.

The reason for taking this roundabout approach to reach our goal is that $\mathcal{H}(M)$, $\mathcal{X}(M)$, etc., are not Banach spaces and the above techniques can not be applied directly. For a different approach which also proves that these spaces have the Baire property see [11].

2.7 Equivalence and Stability for Hamiltonian Dynamical Systems

If X_H and $X_K \in \mathcal{X}_H(M)$ we say that X_H and X_K are \mathbb{H} -equivalent if there exists a homeomorphism $\phi : M \longrightarrow M$ mapping orbits of X_H onto orbits of X_K and preserving sense.

We say $X_H \in \mathcal{X}_H(M)$ is \mathbb{H} -stable if there exists a neighbourhood θ of X_H in $\mathcal{X}_H(M)$ with the C^1 topology such that $X_K \in \theta$ implies that X_H and X_K are \mathbb{H} -equivalent. This is a very strong type of stability and the most natural. However Robinson [11] has shown that \mathbb{H} -stable systems are not dense in $\mathcal{X}_H(M)$ when M has dimension > 4 . We show later that the \mathbb{H} -stable systems are open-dense in $\mathcal{X}_H(M)$ when M is compact and has dimension 2.

We now consider a very weak type of stability. If X_H and $X_K \in \mathcal{X}_H(M)$ we say that X_H and X_K are \mathbb{E} -equivalent if there exists a diffeomorphism $\phi : M \longrightarrow M$ mapping energy surfaces of H onto energy surfaces of K (and hence critical points onto critical points also).

We say $X_H \in \mathcal{X}_H(M)$ is \mathbb{E} -stable (Energy stable) if there exists a neighbourhood θ of X_H in $\mathcal{X}_H(M)$ with the C^1 topology such that

$X_K \in \theta$ implies that X_H and X_K are \mathbb{E} -equivalent. This is a very weak form of stability since it takes no account of the flow on each energy surface and is independent of the symplectic form associated with M . However it is a useful type of stability for two reasons:

a) We would expect \mathbb{E} -stability to be implied by any stronger form of stability. We show below that the set of \mathbb{E} -stable systems are open-dense in $\mathcal{X}_H(M)$ when M is a compact symplectic manifold. Thus, our expectation can be satisfied in this case without weakening any openness or density results we may obtain.

b) Conversely if we do not have \mathbb{E} -stability in a neighbourhood we should not expect any stronger type of stability to hold.

Note that the above definitions of equivalence and stability extend naturally to $\mathcal{X}_{LH}(M)$.

2.8 Graphs

Let $I = [0,1]$, $\overset{\circ}{I} = (0,1)$, and $\partial I = \{0\} \cup \{1\}$.

A graph G is a Hausdorff topological space with structure $G = V \cup E$, where

- 1) $V = \bigcup_{j \in J} v_j$ is a union of (distinct) points v_j called vertices.
- 2) $E = \bigcup_{k \in K} e_k$, and for each edge e_k there exists a surjection $h_k : I \longrightarrow e_k$ with
 - a) $h_k|_{\overset{\circ}{I}}$ is an embedding for all k ,
 - b) $h_i(\overset{\circ}{I}) \cap h_k(\overset{\circ}{I}) = \emptyset$ for all $i \neq k$,
 - c) $h_k^{-1}(V) = \partial I$ for all k .

We will always assume that the indexing sets J and K are countable. If J and K are both finite sets we say that G is a finite graph.

We say that an edge e_k is incident with a vertex v_j (and vice-versa) if $e_k \cap v_j \neq \emptyset$. Thus every edge is incident with either one or two vertices. We say two edges are adjacent if they are incident with a common vertex, and two vertices are adjacent if they are incident with a common edge.

A graph G is locally finite if the number of edges incident with a vertex v_j is finite for every j . We will usually assume that graphs are locally finite.

Let C_k be the set of surjective maps $h: I \longrightarrow e_k$,

with $h|I$ an embedding. If $h, h' \in C_k$ we define an equivalence relation $h \sim h'$ if $h|_{\partial I} = h'|_{\partial I}$ and h and h' are homotopic keeping ∂I fixed. \sim divides C_k into two equivalence classes called orientations of e_k . Choosing a direction along e_k is equivalent to choosing an orientation of e_k for we can then choose the orientation with representative $h : I \longrightarrow e_k$ for which the point $h(t)$ moves in the direction given as t increases. Conversely, choosing an orientation gives a direction to the edge.

An oriented graph is a graph G together with an orientation for each edge e_k of G .

If e_k is an edge of an oriented graph G and h is a representative of the orientation of e_k , then we call $v_i = h(0)$ the initial vertex of e_k and $v_j = h(1)$ the terminal vertex of e_k . We say that e_k is oriented away from v_i and towards v_j .

A path in an (oriented) graph G is a finite sequence of edges e_1, \dots, e_n and vertices v_0, \dots, v_n , (which we will also write $v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n$) not necessarily all distinct, such that e_j is incident with v_{j-1} and v_j for $j = 1, \dots, n$.

The path is oriented if each edge e_i is oriented away from v_{i-1} towards v_i . An arc is a path in which all the v_i , except possibly v_0 and v_n , are distinct. A circuit is an arc with $v_0 = v_n$. A closed path is a path with $v_0 = v_n$.

Two paths p_1 and p_2 in G are said to be equivalent if we can obtain p_2 from p_1 by successively removing or inserting in p_1 , at appropriate points, an edge traversed successively in each direction. i.e. an operation

$$(v_0, e_1, \dots, e_i, v_i, e_{i+1}, \dots, e_n, v_n) \longrightarrow (v_0, e_1, \dots, e_i, v_i, e_k, v_l, e_k, v_i, e_{i+1}, \dots, e_n, v_n)$$

or the reverse.

Alternatively, we can give a topological definition of a path, viz. a path p in G is a mapping $p : I \longrightarrow G$ with $p^{-1}(v)$ a union of isolated points containing 0 and 1 and such that p is a local homeomorphism on the interior of each edge. We identify p with p' if they are homotopic by a homotopy P_t , $0 \leq t \leq 1$, keeping 0 and 1 fixed and such that each P_t is a path in G . We leave it to the reader to verify that this is a valid definition and equivalent to the first. With this

definition, two paths p and p' are equivalent if they are homotopic keeping 0 and 1 fixed.

Let G and H be graphs with indexing sets J_G , K_G , and J_H , K_H resp. A homomorphism from G into H is a pair of maps $f_1 : K_G \longrightarrow K_H$ and $f_2 : J_G \longrightarrow J_H$ such that if an edge e_k is incident with a vertex v_j in G then $e_{f_1(k)}$ is incident with $v_{f_2(j)}$ in H . If G and H are oriented graphs we demand also that the maps (f_1, f_2) preserve orientation consistently or reverse orientation consistently. Given a homomorphism of G into H we can find a continuous mapping $f : G \longrightarrow H$ which maps vertices to vertices, edges to edges, and consistently preserves or reverses orientation if necessary. Conversely such a mapping induces a homomorphism $(f_1, f_2) : G \longrightarrow H$. A 1-1 homomorphism of G onto H is called an isomorphism of G with H . In this case we can find a homeomorphism of G with H induced by the isomorphism.

An oriented homomorphism $G \longrightarrow H$ of oriented graphs is a homomorphism preserving orientation. Similarly an oriented isomorphism.

If G is an oriented graph, we have a reflexive, transitive relation \leq defined on the points of G by $x \leq y$ if there exists an oriented path p in G , $p : I \longrightarrow G$, so that $x = p(t_1)$, $y = p(t_2)$, and $t_1 \leq t_2$.

Suppose that $x \leq y$ and $y \leq x$. If $x \neq y$ it is clear that there exists an oriented path p in G from x to y , and also an oriented path p' from y to x . Hence there exists an oriented circuit in G . Conversely, if G contains no oriented circuit then $x \leq y$ and $y \leq x$ implies that $x = y$, and then \leq is a partial order on the points of G . Its restriction to the vertices V of G is a partial order on V .

For every pair (v_i, v_j) of vertices of G , define $V_{ij} = \{v_k \in V : v_i \leq v_k \leq v_j\}$. If V_{ij} is a finite set for every pair (v_i, v_j) , then we say that G is finitely connected. If G is finitely connected then there is a finite upper bound to the number of edges in an oriented arc from v_i to v_j , for each pair (v_i, v_j) , and there are finitely many such arcs.

We can now prove the propositions which we will need later. Let $x < y$ denote $x \leq y$ and $x \neq y$.

Proposition 1 Suppose that there exists a function $f' : V \longrightarrow \mathbb{Z}$ on the vertices of the oriented graph G such that $f'(v_1) < f'(v_2)$ whenever $v_1 < v_2$. Then we can extend f' to a continuous function $f : G \longrightarrow \mathbb{R}$ with $f(x) < f(y)$ whenever $x < y$.

Proof Extend the function defined on the vertices incident with an edge e_k to the whole edge e_k by choosing some map $h_k : I \longrightarrow e_k$ and extending linearly w.r.t. the induced linear structure on e_k . ||

Proposition 2 Suppose that G is a finitely connected locally finite graph containing no oriented circuits. Then there exists a function $f : G \longrightarrow \mathbb{R}$ such that $f(V) \subset \mathbb{Z}$ and $f(x) < f(y)$ when $x < y$.

Proof W.l.o.g. we may suppose that G is connected, for if not we deal with each component separately. Choose an ordering v_0, v_1, \dots for the vertices of G as follows: Choose v_0 . Now order the vertices adjacent to v_0 as v_1, \dots, v_{n_1} . Take v_1 and order the vertices adjacent to v_1 and not already ordered (if any) as $v_{n_1+1}, \dots, v_{n_2}$. Take v_2 , etc. Since G is connected and locally finite this process orders all the vertices of G .

We now define a sequence of functions

$f_1 : V \longrightarrow \mathbb{Z}$. Define $f_0(v_i) = i$. If $v_1 < v_0$ define $f_1(v_0) = 0$, $f_1(v_1) = -1$, and $f_1(v_i) = i-1$ for $i \neq 0,1$. Otherwise define $f_1 = f_0$.

Now suppose that we have defined $f_n : V \longrightarrow \mathbb{Z}$ with the following properties:

1_n) f_n is an injection and takes successive integer values.

2_n) f_n takes its lowest $n+1$ values on $V_n = \{v_0, \dots, v_n\}$.

3_n) $f_n|_{V_n}$ satisfies the hypothesis of

Proposition 2.8.1 and $f_n|_{V-V_n} = f_0|_{V-V_n} - \text{const.}$

4_n) $f_n(v_0) = 0$

5_n) Let $f_n(v_{n+1}) = m$. Define

$A_{nk} = \{f_n(v_j) : v_k < v_j \text{ and } f_n(v_j) < m\}$, and

$B_{nk} = \{f_n(v_j) : v_j < v_k \text{ and } f_n(v_j) < m\}$, for $0 \leq k \leq n$.

If $A_{nk} \neq \emptyset$ then it has a least member a_{nk} ; define

$p_{nk} = \min\{j : a_{nk} \leq f_n(v_j) < m\}$. If $B_{nk} \neq \emptyset$ then it

has a greatest member b_{nk} ; define $q_{nk} = \min\{j :$

$f_n(v_j) \leq b_{nk}\}$. Then if $f_n(v_k) > 0$ and $A_{nk} \neq \emptyset$ we

have $v_k < v_{p_{nk}}$, and if $f_n(v_k) < 0$ and $B_{nk} \neq \emptyset$ we

have $v_{q_{nk}} < v_k$. (End of 5_n) .

We now define f_{n+1} . Let $A_n = A_{n,n+1}$ and $B_n = B_{n,n+1}$. If $A_n = \emptyset$ we define $f_{n+1} = f_n$. If $B_n = \emptyset$ define

$$\begin{aligned} f_{n+1}(v_{n+1}) &= m - n - 2, \\ f_{n+1}(v_i) &= f_n(v_i), \text{ for } v_i \text{ with } f_n(v_i) < m, \\ f_{n+1}(v_i) &= f_n(v_i) - 1, \text{ for } v_i \text{ with } f_n(v_i) > m. \end{aligned}$$

By the construction of the ordering of the vertices we cannot have both $A_n = \emptyset = B_n$.

If $A_n \neq \emptyset \neq B_n$, then A_n has a least member a_n and B_n has a greatest member b_n , and $b_n < a_n$.

Let $p_n = \min\{i : a_n \leq f_n(v_i) < m\}$ and $q_n = \min\{i : f_n(v_i) \leq b_n\}$. Then $0 \leq p_n, q_n \leq n$, and $p_n \neq q_n$.

If $p_n < q_n$ define

$$\begin{aligned} f_{n+1}(v_i) &= f_n(v_i) - 1, \text{ for } v_i \text{ with } f_n(v_i) \leq b_n, \\ f_{n+1}(v_{n+1}) &= b_n, \\ f_{n+1}(v_i) &= f_n(v_i), \text{ for } v_i \text{ with } b_n < f_n(v_i) < m, \\ f_{n+1}(v_i) &= f_n(v_i) - 1, \text{ for } v_i \text{ with } m < f_n(v_i). \end{aligned}$$

Note that since $p_n < q_n$, $q_n > 0$, so by 4_n $b_n < 0$.

If $p_n > q_n$ define

$$\begin{aligned} f_{n+1}(v_i) &= f_n(v_i), \text{ for } v_i \text{ with } f_n(v_i) < a_n, \\ f_{n+1}(v_{n+1}) &= a_n, \\ f_{n+1}(v_i) &= f_n(v_i) + 1, \text{ for } v_i \text{ with } a_n \leq f_n(v_i) < m, \\ f_{n+1}(v_i) &= f_n(v_i), \text{ for } v_i \text{ with } m < f_n(v_i). \end{aligned}$$

Note that since $p_n > q_n$, $a_n > 0$.

Finally, if $V = \{v_0, \dots, v_n\}$ define $f_r = f_n$ for all $r > n$.

We must check that f_{n+1} satisfies conditions $1_{n+1}, \dots, 5_{n+1} \cdot 1_{n+1}$ to 4_{n+1} are easy. We split 5_{n+1} into several cases:

- a) $A_n = \emptyset$. It is sufficient to note that $A_{n+1, n+1} = A_n = \emptyset$, $B_{n+1, n+1} = B_n$, and if $0 \leq k \leq n$ then $A_{n+1, k}$ is equal to A_{nk} or $A_{nk} \cup \{m\}$ and $B_{n+1, k} = B_{nk}$. Thus 5_{n+1} follows from 5_n .
- b) $B_n = \emptyset$. We have $B_{n+1, n+1} = B_n = \emptyset$, $B_{n+1, k}$ is equal to B_{nk} or $B_{nk} \cup \{m-n-2\}$ and $A_{n+1, k} = A_{nk}$ for $0 \leq k \leq n$. Again 5_{n+1} follows from 5_n .
- c) $A_n \neq \emptyset \neq B_n$, $p_n < q_n$. For $v_k \neq v_{n+1}$ we have $A_{n+1, k} = A_{nk}$ whenever $f_{n+1}(v_k) > 0$, so in this case 5_{n+1} follows from 5_n . $B_{n+1, k} \neq B_{nk}$, but we must have $q_{n+1, k} = q_{nk}$ or $q_{n+1, k} = n+1$ and the latter only when $B_{nk} = \emptyset$. In the first case 5_{n+1} follows from 5_n , and in the second case $v_{n+1} < v_k$ and v_{n+1} is the only such element in V_{n+1} . Hence 5_{n+1} . For v_{n+1} we have $f_{n+1}(v_{n+1}) < 0$ so we need only consider $B_{n+1, n+1}$. Let v_i be the element with $f_{n+1}(v_i) = b_n - 1$. Then either $q_{n+1, n+1} = i$ or $q_{n+1, n+1} = q_{ni}$, and in either case we get the required result.

d) $A_n \neq \emptyset \neq B_n$, $p_n > q_n$. This case is similar to (c).

Thus we can define inductively a sequence of functions $\{f_i\}$ satisfying conditions 1_i to 5_i .

We define a function $f : V \longrightarrow \mathbb{Z}$ by setting $f(v_i) = \lim_{n \rightarrow \infty} f_n(v_i)$. To show that this limit

exists it is sufficient to show that given v_i there exists $N = N(i)$ so that $f_n(v_i) = f_N(v_i)$ for all $n \geq N$. It then follows easily that f satisfies the hypotheses of Proposition 2.8.1 and so can be extended to a mapping $f : G \longrightarrow \mathbb{R}$, thus proving this proposition.

To show that the above limit exists we first note that $f_n(v_0) = 0$ for all n . Now choose v_i . Since G is locally finite and finitely connected there exists N such that V_N contains all vertices on oriented paths from v_k to v_j with $v_k, v_j \in V_i$. Let us examine f_{N+1} . v_{N+1} is not on any oriented path from v_k to v_j with $v_k, v_j \in V_i$. If $A_N = \emptyset$ or $B_N = \emptyset$ then $f_{N+1}|_{V_i} = f_N|_{V_i}$ from the definition of f_{N+1} . Suppose that $A_N \neq \emptyset \neq B_N$, and $p_N < q_N$, $q_N > i$, or $p_N > q_N$, $p_N > i$. Then again $f_{N+1}|_{V_i} = f_N|_{V_i}$, so it remains to deal with the cases $p_N < q_N \leq i$ and $q_N < p_N \leq i$. Suppose the former holds.

Then $f_{N+1}(v_{N+1}) < 0$, and by \mathfrak{S}_{N+1} there exists $v_j \in V_i$ with $v_j < v_{N+1}$. Therefore v_{N+1} is not less than v_k for any $v_k \in V_i$. So if $f_N(v_k) \in A_{N+1, N+1} = A_N$ we must have $f_N(v_k) < 0$, or $f_N(v_k) > 0$ and $p_{N+1, k} > i$, and $k > i$. But if there exists v_h with $0 < f_{N+1}(v_k) < f_{N+1}(v_h) < a_{N+1, k}$ then $h > k$. Hence $p_{N+1, k} = p_N < i$ by hypothesis. So $f_N(v_k) < 0$. i.e. we have $v_{N+1} < v_k$ and $f_{N+1}(v_{N+1}) < f_{N+1}(v_k) < 0$. For v_k we have either $B_k = \emptyset$ or $p_k < q_k$ and as $v_{N+1} < v_k$ we have $q_k \leq i$. Therefore $B_k \neq \emptyset$ and we must have $p_k < q_k \leq i$. Hence $p_{N+1, k} < q_{N+1, k} \leq i$ and we can apply the same argument to v_k . Thus we can find an infinite chain v_{N+1}, v_k, \dots with $f_{N+1}(v_{N+1}) < f_{N+1}(v_k) < \dots < 0$. But this is a contradiction and hence $p_N < q_N \leq i$ cannot hold. Similarly we cannot have $q_N < p_N \leq i$.

We have proved that $f_{N+1}|_{V_i} = f_N|_{V_i}$. Inductively we can now show that $f_{N+n}|_{V_i} = f_N|_{V_i}$ for all $n > 0$. Hence $f(v_i)$ is well defined and the proposition follows. ||

Corollary If G is a finite graph containing no oriented circuits, then there exists a function $f : G \longrightarrow \mathbb{R}$ with $f(V) \subset \mathbb{Z}$ and $f(x) < f(y)$ whenever $x < y$.

Chapter 3

Hamiltonian Foliations on Compact Manifolds

In this chapter we will look at the \mathbb{E} -equivalence properties of Hamiltonian dynamical systems on a compact symplectic manifold w.b. The main tool in this study will be the Hamiltonian foliation, defined below, which is a generalisation of the concept of a Hamiltonian function on a manifold w.b. In the techniques using Hamiltonian foliations the symplectic structure is irrelevant and we need only suppose that we are working on a compact manifold w.b. The results which we get can then be applied to the special case of a Hamiltonian dynamical system, but also have applications to the study of functions on manifolds.

In the first section of this chapter we look at the structure of the non-degenerate Hamiltonian dynamical systems. We then use their properties to define a Hamiltonian foliation and note that the concept of \mathbb{E} -equivalence extends naturally to Hamiltonian foliations. In section 3.3 we develop

the graph of a Hamiltonian foliation, which is an invariant under \mathbb{E} -equivalence of Hamiltonian foliations. In section 3.5 we find a complete set of invariants for an \mathbb{E} -equivalence class of Hamiltonian foliations. We then apply the results obtained to Hamiltonian dynamical systems. Finally we look at the \mathbb{E} -stability properties of Hamiltonian dynamical systems and show that an open-dense set in $\mathcal{X}_H(M)$ consists of \mathbb{E} -stable systems.

3.1 Structure of Hamiltonian systems

We will be interested in two subsets of $\mathcal{X}_H(M)$ when M is a compact symplectic manifold w.b.

a) $\mathcal{X}_N(M)$, the set of non-degenerate Hamiltonian systems.

$\mathcal{X}_N(M) = \{X_H \in \mathcal{X}_H(M) : H \in \mathcal{H}_N(M)\}$. This is the largest set of systems for which we can conveniently construct the invariants.

b) $\mathcal{X}_S(M)$, the set of \mathbb{E} -stable Hamiltonian systems.

$\mathcal{X}_S(M) = \{X_H \in \mathcal{X}_N(M) : \text{if } m \text{ is a critical point of } X_H \text{ then the closed energy surface containing } m \text{ contains precisely one critical point, viz. } m\}$

We show in the last section of this chapter that

such systems are \mathbb{E} -stable.

Proposition 1 Each of the sets $\mathcal{X}_N(M)$, $\mathcal{X}_S(M)$, is open-dense in $\mathcal{X}_H(M)$ with the C^r topology, $r \geq 1$, when M is a compact manifold w.b.

Proof Transversality theory, [7]. If M is a closed manifold then the result is elementary. If not then it is more convenient to look at the associated sets of functions on M . It is then an easy consequence of the transversality theory that the set of non-degenerate (resp. stable) functions is open-dense in $\mathcal{F}(M)$. Hence the set of non-degenerate (resp. stable) Hamiltonian functions is open-dense in $\mathcal{H}(M)$. Finally the image of $\mathcal{H}_N(M)$ (resp. $\mathcal{H}_S(M)$ = set of stable Hamiltonian functions) under \mathcal{X} is open-dense in $\mathcal{X}_H(M)$. ||

Let N be a closed manifold. We define the N annulus, A_N , to be the product manifold $N \times [0,1]$. A_N is a manifold w.b. and $\partial A_N = N \times \{0\} \cup N \times \{1\}$. We call these two components the boundaries of A_N , and we call $N \times (0,1) = \overset{\circ}{A}_N$ the open N annulus. Suppose that $\varphi : \overset{\circ}{A}_N \longrightarrow M$ is an embedding in a manifold w.b. M which has an extension

$\bar{\varphi} : A_N \longrightarrow M$ where $\bar{\varphi}$ is an immersion. Then we shall call the sets $\bar{\varphi}(N \times \{0\})$ and $\bar{\varphi}(N \times \{1\})$ the boundaries of A_N (or \mathring{A}_N) in M . Note that the boundaries of A_N in M are not necessarily distinct.

Recall the results on energy surfaces from chapter 2. We prove in addition:

Theorem 2 Let $X_H \in \mathcal{X}_N(M)$. Let $K = \{x : x \text{ is a critical point of } H \text{ or } x \text{ is on a singular energy surface of } H\}$, and $L = \{x : x \text{ is a point on a regular energy surface of } H\} = M - K$. Then L is a finite disjoint union of open annuli, and L is dense in M .

Proof We first note that K is a finite union of isolated critical points, manifolds w.s. of codimension 1 associated to critical points, and boundary components of M . Hence L is open and dense in M .

Let Σ_e be a regular energy surface of H and choose a Riemannian metric on M . Let its inverse gradient have flow φ , and let

$$T = \{t : \varphi_t(m) \text{ is defined for all } m \in \Sigma_e \text{ and } \varphi_t(m) \notin \partial M\}.$$

Then T is an open interval and $A = \bigcup_{t \in T} \varphi_t(\Sigma_e)$ is an open annulus contained in M . If we can show that for each such open annulus A the boundaries of A are contained in K , then the theorem is proved.

Now $\varphi_t(m)$ is defined everywhere in $M - C$, so each boundary of A must contain a critical point or boundary point of M . But by continuity H is constant on a boundary of A . Therefore the whole boundary is contained in an isolated critical point, a manifold w.s. associated to a critical point, or a boundary component of M . Thus each boundary of A is contained in K . ||

Corollary 1 Each component of L is diffeomorphic to an open annulus $\Sigma_e \times (0,1)$ with energy surfaces $\Sigma_e \times \{t\}$, for some regular energy surface Σ_e in M . ||

Corollary 2 If Q is a component of ∂M , B is a boundary of an open annulus A of L , and $B \cap Q \neq \emptyset$, then $B = Q$ and $Q \cong \Sigma_e$ where $A = \Sigma_e \times (0,1)$.

Proof Look at a tubular neighbourhood of Q . ||

We say that the energy surfaces of X_H are consistent if it is possible to define a vector field on $M - C$ which is everywhere transverse to the energy surfaces of X_H .

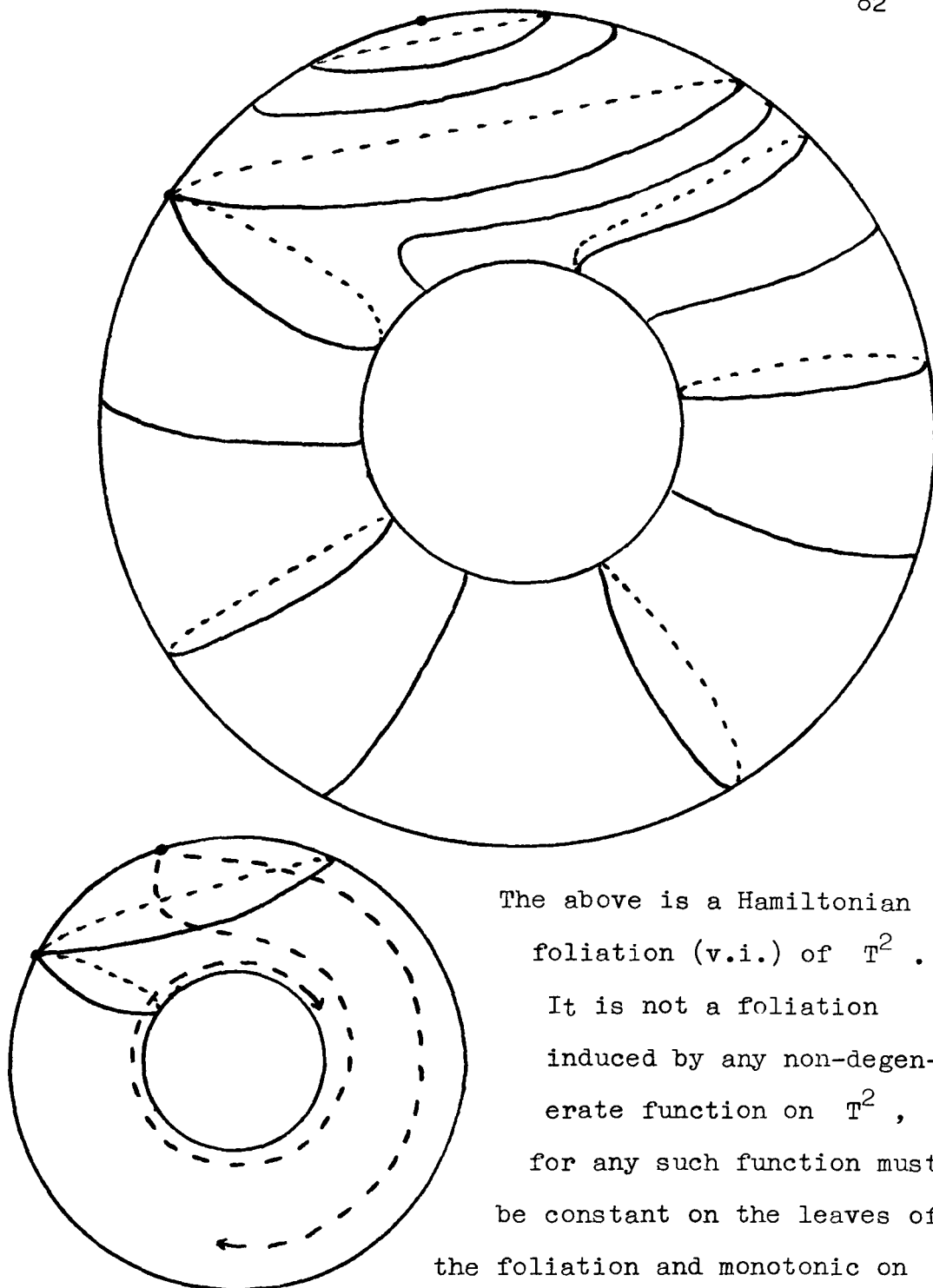
Proposition 3 If $X_H \in \mathcal{X}_H(M)$ then the energy surfaces of X_H are consistent.

Proof The gradient vector field. ||

3.2 Hamiltonian Foliations

In section 1 we derived some obvious properties of Hamiltonian systems. We can now ask if these properties are sufficient to characterise Hamiltonian systems up to \mathbb{E} -equivalence? Elementary examples on T^2 - the two dimensional torus - show that in fact they are not sufficient (see fig. 2) .

We call an object which has the properties of section 1 a Hamiltonian foliation. Note that in the definition below we have simply substituted the term 'leaf' for 'energy surface' used above. In particular any property, such as consistency, defined for energy surfaces of a Hamiltonian system carries over trivially to a property for leaves of a Hamiltonian foliation.



The above is a Hamiltonian
foliation (v.i.) of T^2 .

It is not a foliation
induced by any non-degen-
erate function on T^2 ,
for any such function must

be constant on the leaves of

the foliation and monotonic on

the orbits of a consistent vector field (illustrated left).

Figure 2

Let M be a compact manifold w.b. and let F be a foliation with singular points of M with the following properties:

1) M is foliated by submanifolds of codimension 1, submanifolds w.s. of codimension 1, and isolated points. The singular points of the submanifolds w.s. and the isolated points are called critical points, or singularities, of F .

2) If $m \in M$ is an isolated point of F then there exists a neighbourhood of m which is foliated by spheres of codimension 1 concentric with m .

3) $C = \{x \in M : x \text{ is a critical point of } F\}$ is a finite set satisfying $C \cap \partial M = \emptyset$.

4) $M - C$ is foliated by submanifolds, called leaves of F , which are closed in $M - C$. Those leaves which are closed in M and do not intersect ∂M are called regular leaves. Components of ∂M and the intersections with $M - C$ of submanifolds w.s. of F are called singular leaves. The closure (in M) of a leaf is called a closed leaf.

5) $L = \{x \in M : x \text{ is in a regular leaf of } F\}$ is a finite union of disjoint open annuli, each diffeomorphic to $N \times (0,1)$ with leaves $N \times \{t\}$ for some closed manifold N , and L is dense in M .

6) The foliation is consistent.

We call such a foliation a Hamiltonian foliation of M .

Clearly the energy surfaces of a non-degenerate Hamiltonian function or vector field on M induce a Hamiltonian foliation of M .

We make the following definitions:

$K = M - L = \{x \in M : x \text{ is a point on a singular leaf of } F \text{ or } x \text{ is a critical point of } F\}$.

The connected components of L are denoted $L_{\alpha_1}, \dots, L_{\alpha_r}$. Each component is an open annulus.

The connected components of K are denoted $K_{\beta_1}, \dots, K_{\beta_s}$. Each component is a closed singular leaf, or an isolated critical point.

Note that the concept of \mathbb{E} -equivalence extends naturally to the set of Hamiltonian foliations. However we do not have a natural topology on the set of Hamiltonian foliations so there is no stability property.

Let $\mathcal{HF}(M)$ denote the set of Hamiltonian foliations on M .

3.3 Graph of a Hamiltonian Foliation

Let F be a Hamiltonian foliation on a compact manifold w.b. M . We define an equivalence relation \sim on M by $x \sim y$ if x and y lie in the same closed leaf of F . Let $\Gamma = \Gamma(F)$ denote the quotient space and $q : M \longrightarrow \Gamma$ the quotient map. Γ is called the graph of the foliation F . In the terminology of foliations $\Gamma = M/\text{closed leaves}$.

Theorem 1 $\Gamma(F)$ is a finite connected graph and

- a) q induces a bijection from the set $\{\alpha_1, \dots, \alpha_r\}$ to the set of edges of the graph.
- b) q induces a bijection from the set $\{\beta_1, \dots, \beta_s\}$ to the set of vertices of the graph so that the edge corresponding to α_j is incident with the vertex corresponding to β_k if and only if $\bar{L}_{\alpha_j} \cap K_{\beta_k} \neq \emptyset$.

Proof Note that we are assuming throughout this chapter that M is compact and connected.

We first prove that $\Gamma(F)$ is a graph. In fact we assert that the edges of the graph are the images under q of the sets $\bar{L}_{\alpha_1}, \dots, \bar{L}_{\alpha_r}$, and the vertices of the graph are the images under q of $K_{\beta_1}, \dots, K_{\beta_s}$. Thus it is sufficient to show

that for each a_i , there exists a surjection $h_i : I \longrightarrow q(\bar{L}_{u_i})$ satisfying the conditions of the definition (section 2.8).

Now $L_{u_i} \cong N_i \times \overset{\circ}{I}$ for some closed manifold N_i . Choose a point $x \in N_i$ and define $p_i' : \overset{\circ}{I} \longrightarrow L_{u_i}$ by $t \longmapsto (x, t)$. Then we can extend p_i' uniquely to a map $p_i : I \longrightarrow M$ and $p_i(I) \subset \bar{L}_{u_i}$. Define $h_i = q \circ p_i : I \longrightarrow \Gamma$. Then h_i is a surjection of I onto $q(\bar{L}_{u_i})$ and satisfies:

- a) $h_i|_{\overset{\circ}{I}}$ is an imbedding,
- b) $h_i(\overset{\circ}{I}) \cap h_j(\overset{\circ}{I}) = \emptyset$ for all $i \neq j$. This is clear as the annuli are disjoint.
- c) $h_i^{-1}(V) = \partial I$, where V is the set of vertices of Γ . This follows from the observation that $p_i(\overset{\circ}{I}) \subset L_{u_i}$ and $p_i(\partial I)$ is contained in the boundaries of L_{u_i} in M , and these are contained in K .

That the graph is connected follows from the connectedness of M . That the graph is finite follows from the compactness of M , for the critical points of F form a set of isolated points in M and hence a finite set of points. Also ∂M has finitely many components, and since each component of K is either a component of ∂M or contains at least one critical point, Γ can have at most a

finite number of vertices. Similarly if we take a sufficiently small open neighbourhood U of K then U and the components of L form an open cover of M and no smaller collection of these sets cover M . Hence we must have a finite cover of M . Thus Γ has finitely many edges also. That the map q induces bijections follows at once from the earlier part of the proof. ||

If the Hamiltonian foliation is induced by a non-degenerate function H , we can state the equivalence relation as: $x \sim y$ if and only if $H(x) = H(y) = e$ and x and y are in the same connected component of $H^{-1}(e)$.

We define an orientation for the graph Γ as follows: Choose an edge α_i of Γ and choose an orientation for α_i . If Γ has any other edges then it has an edge α_j and a common vertex β_k of α_i and α_j such that $\bar{U}_{\alpha_i} \cap \bar{U}_{\alpha_j} \cap K_{\beta_k}$ contains a point x which is not a critical point. Choose such an α_j and β_k , and if α_i is oriented towards (resp. away from) β_k orient α_j away from (resp. towards) β_k . Now if Γ has any other edges it has one α_h adjacent to either α_i or α_j and a common vertex β_l with α_j , say, such that

$\bar{L}_{a_i} \cap \bar{L}_{a_h} \cap K_{\beta_1}$ contains a point y which is not a critical point, etc. We assert that this process orients the graph.

Theorem 2 The process described above gives an orientation for Γ which is unique up to isomorphism. (i.e. consistent reversing of orientation is possible)

Proof We prove that having chosen the orientation for the first edge a_1 , the orientation for every other edge is uniquely determined by the procedure.

Firstly, we show that the process orients every edge. For suppose that the process orients the edges a_1, \dots, a_i , but that it is impossible to continue the process to orient the edges a_{i+1}, \dots, a_r . Since M is connected there exists $a_j \in \{a_1, \dots, a_i\}$ and $a_k \in \{a_{i+1}, \dots, a_r\}$ such that $\bar{L}_{a_j} \cap \bar{L}_{a_k} \neq \emptyset$. Hence the intersection must be a union of critical points. Let us look at a neighbourhood of one of these. The critical point cannot be an isolated critical point by hypothesis (2) of the definition, and so must have a submanifold w.s. associated to it, the critical point being a singular point. But then it is clear from an examination of a neighbourhood of the critical point that the orientability of a_j implies

the orientability of α_k , either directly or after a further edge α_1 which is adjacent to both α_j and α_k has been oriented.

Secondly, we show that each edge has a uniquely defined orientation. Suppose not, i.e. that there exists an edge α_k and by appropriate choice of edges we can induce either orientation on α_k . Thus there exist two sequences of edges $\alpha_1, \alpha_{i_1}, \dots, \alpha_{i_n}, \alpha_k$ and $\alpha_1, \alpha_{j_1}, \dots, \alpha_{j_m}, \alpha_k$ inducing opposite orientations on α_k . We will show that this contradicts the consistency of the foliation.

We can construct a vector field on L_{α_1} , transverse to the foliation at each point, such that if v_x is the vector at $x \in L_{\alpha_1}$ then the induced vector at $q(x) \in \alpha_1$ is a vector in the direction in which α_1 is oriented. Now this vector field can be smoothly extended to $\bar{L}_{\alpha_1} - C$ and thence to $L_{\alpha_{i_1}}$ so that at each point where it is defined it is transverse to the foliation. Notice that by our construction, if $x \in L_{\alpha_{i_1}}$ then the induced vector at $q(x) \in \alpha_{i_1}$ is again in the direction in which α_{i_1} is oriented. Similarly we can extend the vector field to $L_{\alpha_{i_2}}, \dots, L_{\alpha_k}$, and to $L_{\alpha_{j_1}}, \dots$

L_{u_k} . But on L_{u_k} the two extensions are in opposite directions transverse to the foliation, and so it is impossible to define a vector field everywhere transverse to the foliation. Contradiction. ||

If the Hamiltonian foliation is induced by a Hamiltonian function H then we have an induced function h on the graph, defined by

$$h(x) = H(q^{-1}(x))$$

the right hand side being well defined. Now in the oriented graph it is easily seen that the orientations of the edges denote consistently either the direction in which h is increasing or the direction in which h is decreasing. W.l.o.g. we will always suppose that the graph is oriented so that the former holds.

Proposition 3 If two Hamiltonian foliations of M are \mathbb{E} -equivalent then their oriented graphs are isomorphic. ||

Corollary If two Hamiltonian dynamical systems on M are \mathbb{E} -equivalent then their oriented graphs are isomorphic. ||

3.4 Representative Paths

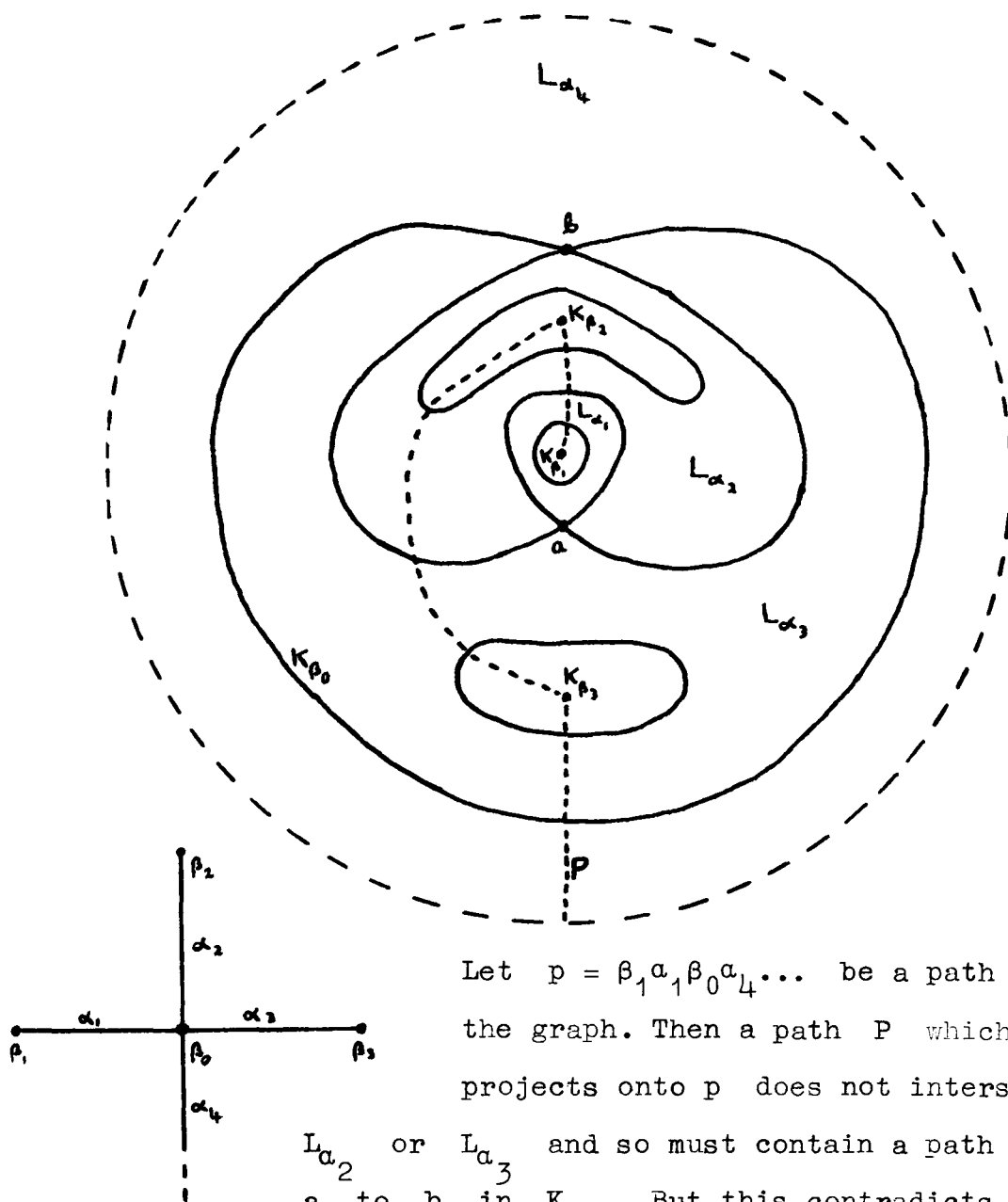
Let F be a Hamiltonian foliation of a compact manifold w.b. M , and let p be a path in $\Gamma = \Gamma(F)$. We wish to construct a topological path p' in M whose projection by q is the path p . However we can construct examples to show that this is not always possible (see fig. 3). It is possible to construct a path p' in M whose projection is a path p_0 in Γ which is equivalent to p (see section 2.8) ., and we call such a path a representative path for p .

Proposition 1 If p is a path in Γ then there exists a representative path for p .

Proof Using the topological definition for p we have $p^{-1}(V) = \{t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1\}$. Put $p(t_i) = \beta_i$ and let the edge of Γ in p joining β_{i-1} to β_i be α_i . Thus p is the path $\beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \beta_{n-1}, \alpha_n, \beta_n$.

Choose a point x_0 in $K_{\beta_0} \cap \bar{L}_{\alpha_1}$, and if possible (i.e. if K_{β_0} is not an isolated critical point) choose $x_0 \notin C$. We must now consider three cases:

- a) if $K_{\beta_1} = \{x_1\}$, choose x_1 ,
- b) if $(\bar{L}_{\alpha_1} \cap \bar{L}_{\alpha_2} \cap K_{\beta_1}) - C \neq \emptyset$, then choose



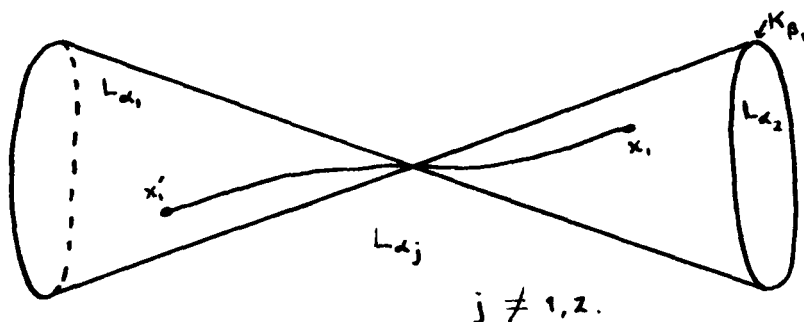
Let $p = \beta_1 \alpha_1 \beta_0 \alpha_4 \dots$ be a path in the graph. Then a path P which projects onto p does not intersect L_{α_2} or L_{α_3} and so must contain a path from a to b in K_{β_0} . But this contradicts the assumption that $p^{-1}(\text{vertices})$ is a union of isolated points. The path P illustrated is a representative path for p and projects onto the path $p' = \beta_1 \alpha_1 \beta_0 \alpha_2 \beta_2 \alpha_2 \beta_0 \alpha_3 \beta_3 \alpha_3 \beta_0 \alpha_4 \dots$ which is equivalent to the path p . Note that P must be 'monotonic' on energy surfaces in each L_{α_j} .

Figure 3

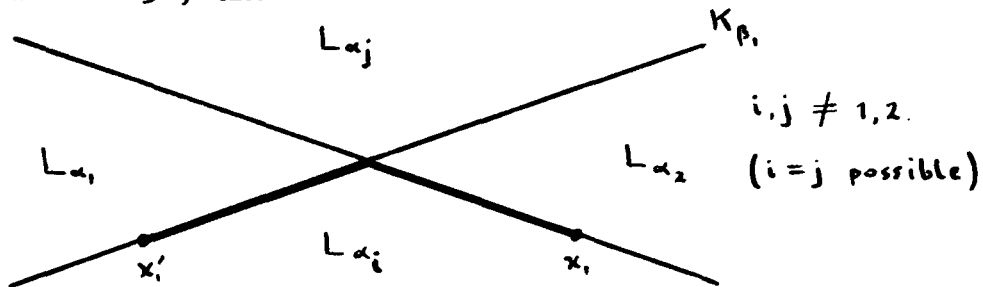
x_1 in this set,

c) if $(\bar{L}_{\alpha_1} \cap \bar{L}_{\alpha_2} \cap K_{\beta_1}) - C = \emptyset$, choose x'_1 in $(\bar{L}_{\alpha_1} \cap K_{\beta_1}) - C$ ($\neq \emptyset$) and x_1 in $(\bar{L}_{\alpha_2} \cap K_{\beta_1}) - C$ ($\neq \emptyset$) and a path in K_{β_1} joining x'_1 and x_1 and avoiding critical points wherever possible.

In the first two cases define the representative path p' on the interval $[0, t_1]$ to be a smooth path with $p'(0) = x_0$, $p'(t_1) = x_1$, joining x_0 to x_1 in \bar{L}_{α_1} , and everywhere transversal to the leaves of the foliation. For the third case we must look at the path joining x'_1 to x_1 . The path must pass through a critical point of F , for if not then x'_1 and x_1 are both in $(\bar{L}_{\alpha_1} \cap \bar{L}_{\alpha_2} \cap K_{\beta_1}) - C$. Since we adopt the same procedure at each point of C on the path, we suppose that the path contains just one critical point. We first observe that the singular point must have singularity index $(n-1, 1)$ in the closed leaf containing it, for otherwise we can perturb the path to avoid the singular point. Furthermore the situation near the critical point must be



for $n \geq 3$, and



for $n = 2$, where n is the dimension of M .

Choose a point y_1 on the other boundary of L_{α_j} . We can now define the representative path p' on $[0, t_1]$ to be a sequence of paths constructed as before and running from x_0 to x'_1 on $[0, t_1/3]$, from x'_1 to y_1 on $[t_1/3, 2t_1/3]$, and from y_1 to x_1 on $[2t_1/3, t_1]$.

It is clear that $q \circ p'|[0, t_1]$ is a path in the graph equivalent to the first edge of p . Similarly we define p' on $[t_1, t_2]$, $[t_2, t_3]$, ... successively continuing the path already defined on $[0, t_1]$. Note that if K_{β_1} is an isolated point then $L_{\alpha_1} = L_{\alpha_2} = S^{n-1} \times (0, 1)$. Hence the end point of partially constructed path is always in the correct position for us to construct the next section.

Clearly the result is a representative path for p . ||

Corollary If p is a closed path in Γ , then we

may choose p' to be a closed path in M .

Proof Follows from the method of construction of p' in the proposition. ||

3.5 A Complete Set of Invariants for a Hamiltonian Foliation

Let M be a compact connected manifold w.b. and F a Hamiltonian foliation of M . Recall that the submanifolds w.s. of the foliation are contained in the set $\{K_{\beta_i}\}$ of closed singular leaves and isolated critical points.

Proposition 1 If M is orientable and K_{β_i} is a submanifold w.s. of the foliation F , then K_{β_i} is an orientable manifold w.s.

Proof We have to prove

- a) that K_{β_i} admits a trivial singular bundle,
- b) that the trivial singular bundle is orientable.

Let $f : U \longrightarrow M$ be a closed tubular neighbourhood for K_{β_i} in M , where $U \subset E$ is a closed neighbourhood of K_{β_i} in E and (π, E, K_{β_i}) is a

singular bundle over K_{β_i} . It is sufficient to prove that E is a trivial singular bundle, for then (b) follows from the fact that M is orientable.

The foliation F is consistent, i.e. we can choose a vectorfield on M which is transverse to the leaves of F . The diffeomorphism f induces a corresponding vector field on U which can then be extended to the whole of E . Now as in Theorem 1.7.3 we may take U so small that the leaves of the foliation are transverse to the fibres of E in U . Hence we can replace the induced vector field by a vector field with vectors everywhere along the fibres of E and having the same direction transverse to the fibres.

Now to show that E is a trivial singular bundle we use a minor variation of the proof of Theorem 1.7.2. Choose a smooth norm $\| \cdot \|$ on E and let $T = \| \cdot \|^{-1}[0,1]$. ∂T is a submanifold of E and can be written $\partial T = \partial T_+ \cup \partial T_-$ where ∂T_+ (resp. ∂T_-) = $\{x \in \partial T : \text{the vector at } x \text{ is an outward (resp. inward) normal to } T\}$. We now proceed precisely as in Theorem 1.7.2. ||

Corollary If K_{β_i} is a submanifold w.s. of the

foliation and $f : U \longrightarrow M$ ($U \subset E$) is a tubular neighbourhood for K_{β_i} in M , then E is a trivial singular bundle over K_{β_i} . ||

Note that the corollary does not depend upon the orientability of M .

Proposition 2 Let N be a manifold w.s., (π, E, N) a singular bundle over N , and H, H' , two smooth functions on E with (for $K = H, H'$) $K^{-1}(0) = N \subset E$, $\partial K_- = N = \partial K_+$, and K has no singularities in $E - N$. Then H and H' are E -equivalent.

Proof The flows for $\text{grin } H$ and $\text{grin } H'$ give singular bundle fibrations of E w.r.t. which H and H' resp. are equal to the standard function constructed in Theorem 1.7.1, or to its negative. Hence by the uniqueness theorem 1.6.2 we have a diffeomorphism $F : E \longrightarrow E$ so that $H' = H \circ F$. ||

We have similar results for functions defined in a neighbourhood of a boundary component of M , or in a neighbourhood of an isolated singular point.

Proposition 3 Suppose that H and H' are two

smooth functions on $N \times [0, \infty)$ where N is a closed manifold, such that (for $K = H, H'$) K has no singularities on $N \times [0, \infty)$ and K is constant on $N \times \{0\}$. Then H and H' are \mathbb{E} -equivalent.

Proof Use the inverse gradient technique of Proposition 2 .

||

Proposition 4 Suppose that H and H' are two smooth non-degenerate functions on \mathbb{R}^n such that (for $K = H, H'$) $K^{-1}(0) = 0 \in \mathbb{R}^n$ and K has no singularities in $\mathbb{R}^n - \{0\}$. Then H and H' are \mathbb{E} -equivalent.

Proof For $K = H, H'$, use the smoothness and non-degeneracy of K to show that in a neighbourhood of 0 K has a power series expansion

$$K(x_1, \dots, x_n) = \frac{1}{2}(x_1^2 + \dots + x_n^2) + \text{higher terms}.$$

Then use the inverse gradient technique of Proposition 2 .

||

Let (π, E, N) be a trivial singular bundle over a manifold w.s. N . In section 1.7 we define a canonical neighbourhood of N in E . On such a

neighbourhood T we use the techniques of section 1.7 to define a function H on T so that $H^{-1}(0) = N \subset T$, $\partial H_+ = N = \partial H_-$, H has no singularities on $T - N$, and $|H|_{\partial T} = 1$. Such a function will be called a canonical function on T , and the foliation it induces on T will be called a canonical foliation of T . T will be called a canonical foliated neighbourhood of N in E .

We make similar definitions for a tubular neighbourhood of $N \times \{0\}$ in $N \times [0, \infty)$. W.l.o.g. we can denote the canonical tubular neighbourhood by $N \times I$. Similarly for a tubular neighbourhood of $0 \in \mathbb{R}^n$. In this case we will take the canonical tubular neighbourhood to be $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

We also extend our definition of surface. We call $N \times \{1\}$ the surface of the tubular neighbourhood $N \times I$, and S^{n-1} the surface of B^n . Note that in neither of these cases do we define positive or negative surfaces.

Theorem 5 Let E be a trivial singular bundle over a manifold w.s. N , and F a Hamiltonian foliation of E whose only closed singular leaf is $N \subset E$ and with no isolated critical points. Then there exists a smooth function H on E satisfying $H^{-1}(0) = N$, $\partial H_- = N = \partial H_+$, and H has no singularities on $E - N$,

such that H is \mathbb{E} -equivalent to the foliation F .

Remark Strictly a Hamiltonian foliation has not yet been defined on a non-compact manifold. We can get around these difficulties here either by looking at a closed tubular neighbourhood of N in E , which is the context in which we will need the result later, or by observing that the only properties we need below are the hypotheses (1) - (6) of the definition of a Hamiltonian foliation and compactness of the closed leaves of F .

Proof E is a trivial singular bundle. Hence using the definition of a foliation H' on E whose energy surfaces are leaves of F , with H' continuous everywhere and smooth on $E - N$, Hence, by Theorem 1.7.3, we can find a smooth function H on E which is \mathbb{E} -equivalent to H' . The result follows. ||

Corollary Let F and F' be any two Hamiltonian foliations of E satisfying the hypotheses of the theorem. Then F and F' are \mathbb{E} -equivalent. ||

Again similar results hold for foliations in the neighbourhood of a boundary component of M , and in the neighbourhood of an isolated singular

point.

We can now define and prove the sufficiency of a complete set of invariants.

A complete set of invariants (C.S.I.) for a Hamiltonian foliation F on M is a pair (X, D) where

$X = \{X_{\beta_i}\}$ is the set of canonical foliated neighbourhoods of closed singular leaves and of isolated singular points of F , and

$D = \{[f_{\alpha_j} : S_p \longrightarrow S_q]\}$ is a set of isotopy classes of diffeomorphisms defined as follows:

Let S_1, \dots, S_m be the set of surfaces of the K_{β_i} .

Then extending the inclusion $K_{\beta_i} \subset M$ we have an inclusion ι_{β_i} , say, of the canonical neighbourhood

X_{β_i} of K_{β_i} into M , which embeds the surfaces of

the X_{β_i} in the annuli L_{α_j} . Now in each L_{α_j}

($= S_{\alpha_j} \times \overset{\circ}{I}$, say) we have precisely two surfaces

S_p and S_q embedded. Suppose that S_p and S_q are embedded as $S_{\alpha_j} \times \{t_0\}$ and $S_{\alpha_j} \times \{t_1\}$ resp.

by ι_{β_h} and ι_{β_i} resp. Let T be the translation

which takes $S_{\alpha_j} \times \{t_0\}$ onto $S_{\alpha_j} \times \{t_1\}$ by the

identity on S_{a_j} . We define $f_{a_j} : S_p \longrightarrow S_q$ by $f_{a_j} = \iota_{\beta_i}^{-1} \circ T \circ \iota_{\beta_h}$. It is clear that the f_{a_j} are defined up to isotopy independently of any choice of embedding. In the set of all f_{a_j} each surface appears precisely once, either as domain or range of some f_{a_j} .

Suppose that we have sets of invariants (X, D) and (X', D') ; that we have a bijection $X \longrightarrow X'$: $X_{\beta_i} \rightsquigarrow X'_{\beta_i}$ and for each β_i a singular bundle isomorphism $x_{\beta_i} : X_{\beta_i} \longrightarrow X'_{\beta_i}$. Then we have an induced bijection between the corresponding sets of surfaces $S_p \rightsquigarrow S'_p$, say, and induced diffeomorphisms $s_p : S_p \longrightarrow S'_p$ which are well defined up to isotopy independently of the embeddings.

We can now prove:

Theorem 6 Let F and F' be Hamiltonian foliations of M with C.S.I. (X, D) and (X', D') resp. Then F and F' are \mathbb{E} -equivalent if and only if there exist bijections $X \longrightarrow X' : X_{\beta_i} \rightsquigarrow X'_{\beta_i}$ and $D \longrightarrow D'$: $[f_{a_j}] \rightsquigarrow [f'_{a_j}]$, and singular bundle isomorphisms $x_{\beta_i} : X_{\beta_i} \longrightarrow X'_{\beta_i}$ for each β_i , such that for all

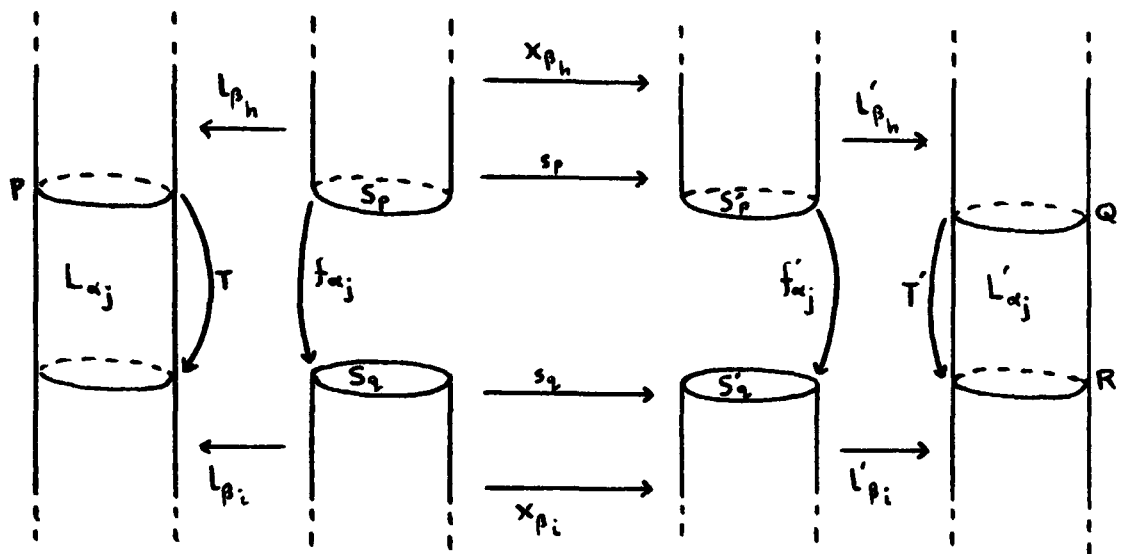
α_j , if $f_{\alpha_j} : S_p \longrightarrow S_q$ then $s_q \circ f_{\alpha_j} \circ s_p^{-1} : S'_p \longrightarrow S'_q$ is isotopic to f'_{α_j} if $f'_{\alpha_j} : S'_p \longrightarrow S'_q$ and to $(f'_{\alpha_j})^{-1}$ if $f'_{\alpha_j} : S'_q \longrightarrow S'_p$.

Remark If the hypotheses of theorem 6 are satisfied then we say that F and F' have the same C.S.I. Thus the above theorem says that two Hamiltonian foliations have the same C.S.I. if and only if they are \mathbb{E} -equivalent.

Proof \longrightarrow . This is clear since the equivalence diffeomorphism provides all the necessary maps.

\longleftarrow . We have inclusions $K_{\beta_i} \subset M$ which we can extend to foliation preserving inclusions of the canonically foliated neighbourhoods X_{β_i} . W.l.o.g. we may suppose that these neighbourhoods have disjoint images in M . Similarly for the $X'_{\beta_i} \subset M$.

For each β_i , the diffeomorphisms $X_{\beta_i} \longrightarrow X'_{\beta_i}$ can be taken to be diffeomorphisms of the canonical neighbourhoods which preserve the foliations. Now if we look at each pair L_{α_j} and L'_{α_j} we have a diffeomorphism induced from the neighbourhood of the boundary of L_{α_j} to a neighbourhood of the boundary of L'_{α_j} as shown:



To prove the result it is sufficient to show that we can extend this diffeomorphism to a foliation preserving diffeomorphism of L_{α_j} onto L'_{α_j} . Now the diffeomorphism of the embedded surface P onto the embedded surface Q is $\iota'_{\beta_h} \circ s_P \circ \iota_{\beta_h}^{-1}$. Thus the induced diffeomorphism from Q to R is

$$\begin{aligned} & \iota'_{\beta_i} \circ s_Q \circ \iota_{\beta_i}^{-1} \circ T \circ \iota_{\beta_h} \circ s_P^{-1} \circ \iota'_{\beta_h}^{-1} \\ &= \iota'_{\beta_i} \circ s_Q \circ f_{\alpha_j} \circ s_P^{-1} \circ \iota'_{\beta_h}^{-1} \\ &\cong \iota'_{\beta_i} \circ f'_{\alpha_j} \circ \iota'_{\beta_h}^{-1} = T' \end{aligned}$$

Hence we can extend the diffeomorphism as required and the theorem is proved. ||

We have proved that any Hamiltonian foliation of a compact manifold w.b. has a canonical

decomposition by canonical foliated neighbourhoods of its closed singular leaves and isolated singular points, and that such a decomposition is sufficient to classify the foliation up to \mathbb{E} -equivalence.

Note that a C.S.I. for a Hamiltonian foliation F gives sufficient information for us to reconstruct the graph directly. For each element X_{β_i} of X we take a vertex β_i for the graph, and for each isotopy class $[f_{\alpha_j}]$ in D we take an edge for the graph, labelled α_j , joining those vertices corresponding to the elements of X whose canonical neighbourhoods contain the domain and range of f_{α_j} . It should be clear that we can also use the canonical neighbourhoods to determine an orientation for the graph.

We now investigate the converse problem: Under what conditions is a pair (X, D) a C.S.I. for some Hamiltonian foliation of some manifold w.b.?

Theorem 7 A necessary and sufficient condition that a pair (X, D) be a C.S.I. for a Hamiltonian foliation F on an n -manifold w.b. M is that

- 1) $X = \{X_{\beta_i}\}$ is a set of canonical foliated

neighbourhoods of a) points,
 or b) compact connected $n-1$ -manifolds,
 or c) compact connected $n-1$ -manifolds
 w.s.

2) $D = \{[f_{\alpha_j}]\}$ is a set of isotopy classes of diffeomorphisms $f_{\alpha_j} : S_p \longrightarrow S_q$, where S_p, S_q , are surfaces of some X_{β_h}, X_{β_i} , in X ; and each such surface occurs precisely once either as domain or range of some f_{α_j} .

3) For each X_{β_i} we can define the surfaces of X_{β_i} to be positive or negative (arbitrarily when K_{β_i} is a point or manifold, and as on p.42 when K_{β_i} is a manifold w.s.) so that each f_{α_j} maps a positive surface to a negative surface or vice versa.

Proof \longrightarrow . This is clear.

\longleftarrow . We will construct the foliated manifold w.b. For each K_{β_i} we take the canonical foliated neighbourhood X_{β_i} and glue these neighbourhoods together using the diffeomorphisms f_{α_j} . Condition (2) ensures that to each surface there corresponds a unique surface to which it is to be glued.

Condition (3) ensures consistency of the foliation. The result is unique up to diffeomorphism and \mathbb{E} -equivalence of the manifold w.b. and the foliation resp., and clearly has (X,D) as its C.S.I. ||

Suppose now that we have a pair (X,D) which we know to be a C.S.I. for some Hamiltonian foliation F of some manifold w.b. M . We would like to know what the manifold w.b. M is without going through the process of constructing it, and in particular we should like some easy criterion to tell us whether two C.S.I. give Hamiltonian foliations on the same manifold w.b. Unfortunately these problems do not appear to have simple solutions. Since the decomposition of a manifold w.b. given by a C.S.I. is similar in conception to a handle decomposition, and can relatively easily be reduced to one, we have a partial solution to the first problem which may in simple cases give a solution to the second problem also. However this procedure is not in general very satisfactory.

3.6 Applications to Hamiltonian Dynamical Systems

Suppose that $X_H \in \mathcal{X}_N(M)$, where (M,ω) is a symplectic manifold w.b. Then we have remarked above that the foliation of M by energy surfaces

of H gives a Hamiltonian foliation of M . Thus:

Theorem 1 Let X_H and $X_K \in \mathcal{X}_N(M)$. Then X_H and X_K are \mathbb{E} -equivalent if and only if they have the same C.S.I.

Proof This is an immediate corollary of Theorem 3.5.6. ||

We should expect the fact that $X_H \in \mathcal{X}_N(M)$ is a Hamiltonian dynamical system to impose some restrictions on the invariants of the induced Hamiltonian foliation. We will investigate these restrictions and the converse problem: Given a Hamiltonian foliation, under what conditions is this foliation \mathbb{E} -equivalent to a foliation induced by a Hamiltonian dynamical system?

Suppose then that we have a Hamiltonian system $X_H \in \mathcal{X}_N(M)$ with C.S.I. (X, D) . We have restrictions on (X, D) due to

- a) the fact that M is a symplectic manifold w.b.,
- b) the fact that the foliation is induced by a function $H \in \mathcal{H}_N(M)$.

As the reader will probably know, the problem of deciding when a given manifold w.b. admits a symplectic form is extremely complex, and no general solution is known. We do know of some restrictions on M , however,

of which the simplest are that M must be even dimensional and orientable. (Others are known, but we will not investigate their effects here.) Thus:

Theorem 2 Let $X_H \in \mathcal{X}_N(M)$ and let K_{β_i} be a connected component of K in the induced Hamiltonian foliation. Then K_{β_i} is either

- a) a point,
 - or b) an odd dimensional orientable manifold,
 - or c) an odd dimensional orientable manifold w.s.,
- and X_{β_i} is resp.
- a) a ball foliated by spheres about K_{β_i} ,
 - or b) a product manifold w.b. $N \times I$ foliated by leaves $N \times \{t\}$, $t \in I$,
 - or c) an orientable trivial singular bundle over K_{β_i} with the canonical foliation.

Proof This is immediately obvious from preceeding results, notably Proposition 3.5.1, part (2) of the definition of a Hamiltonian foliation, Corollary 2 of Theorem 3.1.2, and Proposition 3.5.1. ||

We can also find a condition for M to be an orientable manifold w.b. First we need some notation. Suppose that K_{β_i} is an orientable manifold w.s. in

M , and X_{β_i} is the canonical foliated neighbourhood of K_{β_i} . Let S be any leaf of the foliation. Then S is an orientable manifold and an orientation of S induces naturally an orientation on any other leaf in X_{β_i} by the process of translating an open neighbourhood in S along the fibration. We call such a set of orientations of the leaves of X_{β_i} a leaf orientation of X_{β_i} . Similarly we can define a leaf orientation in the case when K_{β_i} is a point or a manifold.

Theorem 3 Suppose that (X, D) is a C.S.I. for a Hamiltonian foliation of M . In order that M be an orientable manifold it is necessary and sufficient that :

- a) each $X_{\beta_i} \in X$ is a canonical neighbourhood of an orientable manifold or an orientable manifold w.s. or a point,
- b) we can choose a leaf orientation for each of the X_{β_i} in such a way that the maps $f_{\alpha_j} : S_p \longrightarrow S_q$ are orientation preserving diffeomorphisms.

Proof \leftarrow . We define ~~a volume~~ on M as follows:
an orientation

On each X_{β_i} we have an orientation defined using the orientation on each leaf and a vector field on X_{β_i} induced from a consistent vector field on M . (see p.81) Then condition (b) ensures that these orientations are continuous when we glue canonical neighbourhoods together by the f_{α_j} .

→ . Condition (a) is clear from Theorem 2 . To get condition (b) we simply take an orientation on M and reverse the above argument. ||

We can also formulate condition (b) in terms of the graph of the foliation.

Theorem 4 Condition (b) in Theorem 3 is equivalent to the condition:

b') Choose a leaf orientation arbitrarily on each X_{β_i} . Then for every circuit in the graph with edges $\alpha_1, \dots, \alpha_n$, the number of maps in $f_{\alpha_1}, \dots, f_{\alpha_n}$, which are orientation reversing is $\equiv 0 \pmod{2}$.

Proof Given a C.S.I. (X, D) for a Hamiltonian foliation F on M we form the graph Γ of the foliation. Let p be any closed path in the graph, P a closed representative path for p , and p'

the path in the graph onto which P projects. Suppose that p' is the path $\beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \alpha_n, \beta_n = \beta_0$. Corresponding to each edge α_i we have two surfaces, $S_p \subset X_{\beta_{i-1}}$ and $S_q \subset X_{\beta_i}$, and a map $f_{\alpha_i} : S_p \longrightarrow S_q$ (w.l.o.g.). Corresponding to each vertex β_j we have two surfaces $S_t, S_u \subset X_{\beta_j}$ and projecting onto the edges α_j and α_{j+1} , such that given a leaf orientation of X_{β_j} we can find an orientation preserving diffeomorphism g_{β_j} from an open neighbourhood of S_t onto an open neighbourhood of S_u . Note that the maps g_{β_j} are in fact independent of the leaf orientation chosen. We can form the composite map

$$h_p = g_{\beta_n} \circ f_{\alpha_n} \circ g_{\beta_{n-1}} \circ f_{\alpha_{n-1}} \circ \dots \circ g_{\beta_1} \circ f_{\alpha_1}$$

which we may assume to be a well defined map from an open neighbourhood of the domain of f_{α_1} into itself. We call the path p orientation preserving or orientation reversing according as the map h_p is orientation preserving or orientation reversing.

This is well defined independently of the maps chosen and of the representative path chosen for p . That it is independent of the particular maps chosen is clear. Any representative path for p projects onto p' equivalent to p , and so differing from p by a sequence of operations which replace a path

....., β_j , by a path , $\beta_j, \alpha_i, \beta_k, \alpha_i, \beta_j$, or vice versa. Then it is clear that however we choose the leaf orientation on X_{β_k} the difference in the number of orientation reversing maps is either 0 or 2. Hence the orientability of h_p is independent of the representative path chosen for p . By the same argument any null homotopic path in the graph is orientation preserving. Thus we need only check the circuits in the graph.

Now suppose that condition (b) holds. Then we can choose leaf orientations so that all the f_{α_j} are orientation preserving on the leaves of their domain and range. So, using the orientation on M constructed in Theorem 3, we see that glueing the canonical neighbourhoods by the f_{α_j} gives a well defined orientation on the glued neighbourhoods. Hence the orientability of h_p depends on the number of g_{β_j} which, when extended to an open neighbourhood of their domain and range in X_{β_j} , are orientation reversing. But such a map only occurs when the path both enters and leaves X_{β_j} by a positive (or negative) surface, and such a situation corresponds to a vertex β_j of p' for which both α_j and α_{j+1} are oriented towards (or away from) β_j . In any

closed path there are an even number of such vertices. Hence condition (b') holds.

To show that condition (b') implies condition (b) we argue as follows: Choose a circuit p in Γ , let P be a representative path for p in M , and let p' be the path in Γ which P projects onto. Let p' be the path $\beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \alpha_n, \beta_n = \beta_0$. Since condition (b') holds we can choose leaf orientations on each of the X_{β_i} so that the maps f_{α_j} are all orientation preserving. Now take another circuit with at least one edge in common with p . We can choose leaf orientations for all the X_{β_i} involved in this circuit so that

1) all the f_{α_j} involved are orientation preserving, and

2) none of the leaf orientations already fixed for the circuit p are altered.

If this were not possible we should be able to find a circuit in which there were an odd number of orientation reversing f_{α_j} , contradicting condition (b') (we leave the details to the reader). We continue this process for the remaining circuits in the graph. It is then a matter of tidying up loose ends (edges not contained in any circuit) to ensure that we get a leaf orientation satisfying condition (b). ||

We will now look at the restrictions on the invariants imposed by the fact that the foliation is induced by a function $H \in \mathcal{H}_N(M)$.

Proposition 5 Let $H \in \mathcal{H}_N(M)$. Then H induces a Hamiltonian foliation F_H on M and $\Gamma(F_H)$ contains no oriented circuit.

Proof Suppose that we have an oriented circuit $\beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \alpha_n, \beta_n = \beta_0$. Then H induces a function h on the graph (see the remarks following Proposition 3.3.2) and we have

$$h(\beta_0) < h(\beta_1) < \dots < h(\beta_n) = h(\beta_0) .$$

But this is a contradiction. ||

Conversely:

Proposition 6 Let F be a Hamiltonian foliation of M whose graph $\Gamma(F)$ contains no oriented circuits. Then there exists a smooth function H on M inducing the foliation F .

Proof By the Corollary to Proposition 2.8.2 we can define a function h on the graph so that $h(x) < h(y)$ whenever $x < y$, i.e. so that the orientation on each edge denotes the direction in

which h is increasing. Hence we can define a function H_0 on M by $H_0(m) = h(q(m))$ ($q : M \longrightarrow \Gamma$) and we may suppose that H_0 is smooth on each I_{α_j} . But now by Theorem 1.7.3 we can smooth H_0 on each K_{β_i} to get H_1 . H_1 is smooth and \mathbb{E} -equivalent to F , by a diffeomorphism ϕ say. Then $H = H_1 \circ \phi$ is the required function. ||

Thus:

Theorem 7 Let F be a Hamiltonian foliation of a manifold w.b. M . Then a necessary and sufficient condition that F be \mathbb{E} -equivalent to a foliation induced by a function $H \in \mathcal{H}_N(M)$ is that $\Gamma(F)$ contain no oriented circuit. ||

Corollary Let F be a Hamiltonian foliation of a symplectic manifold w.b. (M, ω) . Then a necessary and sufficient condition that F be \mathbb{E} -equivalent to a foliation induced by a Hamiltonian dynamical system is that $\Gamma(F)$ contain no oriented circuit. ||

We will now look at the relation between Hamiltonian foliations and locally Hamiltonian dynamical systems.

Proposition 8 Let (M, ω) be a symplectic manifold $w, b.$ and F a Hamiltonian foliation on M . Then F is \mathbb{E} -equivalent to a locally Hamiltonian dynamical system.

Proof Let F have C.S.I. (X, D) . We may suppose that for each X_{β_i} we have a tubular neighbourhood map of X_{β_i} into M such that the union of the images of the interiors of the X_{β_i} cover M , and the tubular neighbourhood maps are foliation preserving. Choose a vector field on M transversal to the foliation, and on each X_{β_i} choose a canonical function which is increasing in the direction of the (induced) transversal vector field.

Then we have induced on M a set of locally defined functions. Denote the function defined on image X_{β_i} by H_{β_i} . Let U_{ij} be the domain on which both H_{β_i} and H_{β_j} are defined. U_{ij} is contained in the union of annuli $\bigcup L_{\alpha_k}$ and is itself a union of annuli. Thus in a neighbourhood of U_{ij} we can redefine H_{β_i} , say, so that the foliation is unaltered, H_{β_i} is still a smooth function on X_{β_i} , and $H_{\beta_i} - H_{\beta_j}$ is locally constant.

It is clear that the functions H_{p_i} define a locally Hamiltonian dynamical system on (M, ω) and that they induce a foliation \mathbb{H} -equivalent to F .

||

The converse to this proposition is not true. For example, a rational or irrational flow on a torus is locally Hamiltonian (with suitable functions and symplectic form) but does not induce a Hamiltonian foliation. (Problem: Are there any more exceptions, and if so can we classify them?)

Theorem 9 Let X be a locally Hamiltonian dynamical system on the symplectic manifold w.b. (M, ω) . Then a sufficient condition for X to be \mathbb{H} -equivalent to a (globally) Hamiltonian dynamical system is that X induces a Hamiltonian foliation on M whose graph contains no oriented circuits.

Proof Supposing that X induces such a foliation, we can find a function H on M inducing the same foliation and such that the transversal direction in which H increases is the same as that in which the local functions for X increase. Then if $m \in M$ and H_0 is a local function giving rise to X , we can find a neighbourhood U of m and a function f

which is constant on the common energy surfaces of H and H_0 such that $H(x) = f(x) \cdot H_0(x)$ for $x \in U$. But then $X(x)$ and $X_H(x)$ are parallel vectors at each point in U , and thus X and X_H are H -equivalent. ||

Let X be a locally Hamiltonian dynamical system on (M, ω) and suppose that X induces a Hamiltonian foliation of M . Then we may suppose that X is defined locally by functions H_{β_i} (as in the proof of Proposition 8). Let Γ be the graph of the Hamiltonian foliation, and let p be any closed path in Γ . Suppose that P is a representative path for p and p' is the projection of P on Γ . Suppose that p' is the path $\beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \alpha_n, \beta_n = \beta_0$. Then associated to each edge α_i of p' we have a number $n_{\alpha_i} = H_{\beta_i}(x) - H_{\beta_{i-1}}(x)$ where $x \in L_{\alpha_i}$ is a point in the common domain of H_{β_i} and $H_{\beta_{i-1}}$. Define the variation of the path p , v_p , to be the sum
$$v_p = \sum_{i=1}^n n_{\alpha_i}.$$
 Note that the sign of n_{α_i} depends upon the direction in which the edge α_i is traversed. Since any two images of representative paths, p' and p'' say, are equivalent, they differ by a sequence of \vee which replace a operations

path \dots, β_j, \dots by a path $\dots, \beta_j, \alpha_i, \beta_k, \alpha_i, \beta_j, \dots$ or vice versa, we see that v_p is independent of the choice of representative path P and so is well defined. Also v_p is independent of the particular functions H_{β_i} chosen, for each function H_{β_i} occurs in $n_{\alpha_i} = H_{\beta_i}(x) - H_{\beta_{i-1}}(x)$ with positive sign and in $n_{\alpha_{i+1}} = H_{\beta_{i+1}}(y) - H_{\beta_i}(y)$ with negative sign. Now H_{β_i} is defined locally up to an arbitrary constant, and clearly this arbitrary constant has no effect on v_p .

Similarly we see that v_p is independent of the fact that the H_{β_i} are defined on canonical neighbourhoods. For local functions defined on arbitrary neighbourhoods we evaluate v_p by choosing a closed representative path for p , choosing a cover of this path by domains of the local functions, and summing the differences of these functions along the path as before.

Thus for each closed path in the graph we have a well defined variation of the path. It is immediate from the definition that if the closed path p can be decomposed into two disjoint (having no common edge) closed paths p_1 and p_2 , then $v_p = v_{p_1} + v_{p_2}$. Thus the variations of the circuits in Γ determine

the variations of all closed paths in Γ .

Theorem 10 Let X be a non-degenerate locally Hamiltonian dynamical system on the symplectic manifold w.b. (M, ω) . Then a necessary and sufficient condition that X be a (globally) Hamiltonian dynamical system is that X induce a Hamiltonian foliation of M with graph Γ such that $v_p = 0$ for every circuit p in Γ .

Proof Necessity is clear.

Sufficiency: We begin by choosing a locally defined function H_0 generating X . Now if H_1 is another local function generating X whose domain of definition intersects that of H_0 non-trivially then H_0 and H_1 differ by a constant. The condition $v_p = 0$ ensures that this difference is constant even when the common domain is not connected. Thus we can extend H_0 into the domain of H_1 .

We continue this process until the domain of H_0 has been extended to the whole of M . ||

Corollary Suppose that we have a Hamiltonian foliation F of M which is induced by a set of locally defined functions H_i whose difference in a common domain is locally constant. If $v_p = 0$ for all

circuits p in the graph of F , then the foliation is induced by a global function $H : M \longrightarrow \mathbb{R}$ and the graph of F contains no oriented circuits.

Proof $v_p = 0$ for every circuit p in the graph implies that the graph contains no oriented circuits, for if p is an oriented circuit then $v_p > 0$. Hence the foliation is induced by a global function H . ||

Finally we look at the relationship between the homology of a manifold w.b. and the homology of the graph of a Hamiltonian foliation on it.

Let F be a Hamiltonian foliation on a manifold w.b. M , and Γ the graph of F .

Proposition 11 The map $q : M \longrightarrow \Gamma$ induces an epimorphism of homology

$$q_* : H_*(M) \longrightarrow H_*(\Gamma) .$$

Proof We need only check the epimorphism for the 0th and 1st homology groups. For 0th order homology the result is clear, and for the 1st order homology the result follows from the existence of representative paths. ||

Corollary 1 $\dim H_1(\Gamma) \leq \dim H_1(M, \partial M) \leq \dim H_1(M)$. ||

We note that if $\dim H_1(\Gamma) = 0$ then the graph contains no circuits. Hence:

Corollary 2 Suppose that M is a manifold w.b. with $\dim H_1(M) = 0$. Then for any Hamiltonian foliation F of M there exists a smooth function H on M inducing F .

If (M, ω) is a symplectic manifold w.b., then F is \mathbb{E} -equivalent to a foliation induced by a Hamiltonian dynamical system. ||

Corollary 3 Let X be a non-degenerate locally Hamiltonian dynamical system on a symplectic manifold (M, ω) . Then X induces a Hamiltonian foliation of M and X is a (globally) Hamiltonian dynamical system. ||

The final part of the Corollary is an elementary deduction from first principles, and does not in fact need the non-degeneracy qualification.

We remark that if $\dim H_1(M, \partial M) = n$ then the graph of a Hamiltonian foliation on M contains at most n circuits which are independent.

3.7 Stable Hamiltonian Systems

In this section we look at the set of stable Hamiltonian dynamical systems $\mathcal{X}_S(M)$ on a symplectic manifold w.b. (M, ω) . We will eventually prove such systems to be \mathbb{R} -stable, but we begin by looking at the C.S.I. and the graph of a stable Hamiltonian system and show that both are simpler than in the general case.

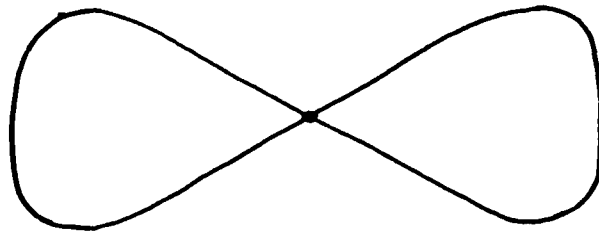
Let F be a stable Hamiltonian foliation of a manifold M . Then if K_{β_i} is a closed singular leaf of F , K_{β_i} has precisely one singularity or is a manifold. In the former case we call K_{β_i} a manifold with one singularity (manifold w.o.s.).

Proposition 1 Let K be a manifold w.o.s. Then the trivial singular bundle (π, E, K) over K is unique up to singular bundle isomorphism.

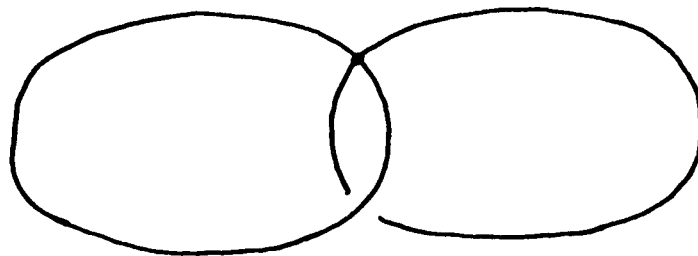
Proof Let m denote the singular point of K . Let U be an open neighbourhood of m in K on which we have local coordinates, and put $V = K - \{m\}$. U and V form an open cover of K . Since V is a manifold we may take the restriction of E to V to be the product $V \times \mathbb{R}$. Note that V may have either

one or two components. The restriction of E to U is isomorphic to the associated neighbourhood $A(U)$. E is defined from these two sets by the transit functions on their intersection. To show that E is unique it is sufficient to show that whenever the transit functions define a trivial bundle, this bundle is isomorphic to the original bundle E . We look at two cases:

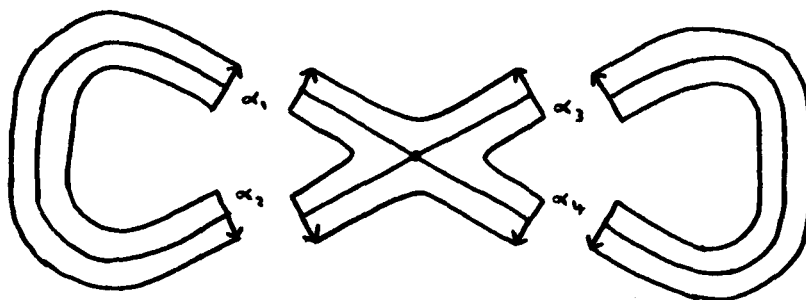
a) K has dimension 1. Then K is either



or

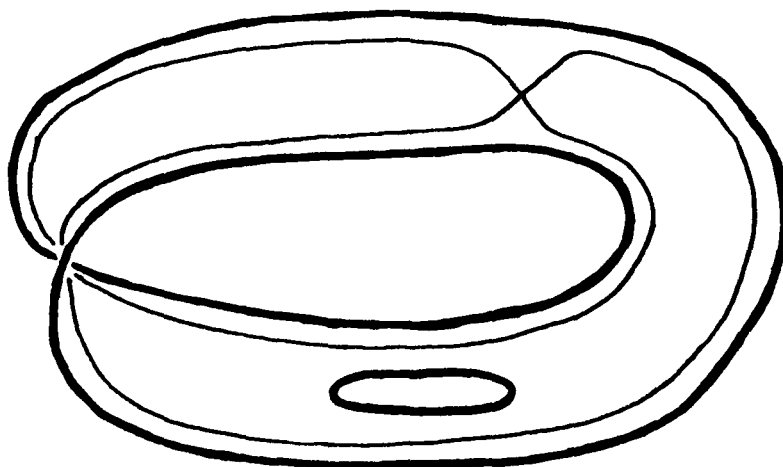


We consider the former case; the latter is similar. The bundle can be represented as



where $\alpha_1, \dots, \alpha_4$ denote the isotopy classes of the transit function restricted to a fibre. Since the bundle is to be trivial we must have $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$. But since the two components of $V \times \mathbb{R}$ are products it is irrelevant how we choose α_1 and α_3 , the resulting bundles are all isomorphic.

Note that in the first case the manifold w.o.s. is orientable and in the second case non-orientable. In fact, in the second case it is not difficult to see that the canonical neighbourhood is a Mobius strip with a disc removed:

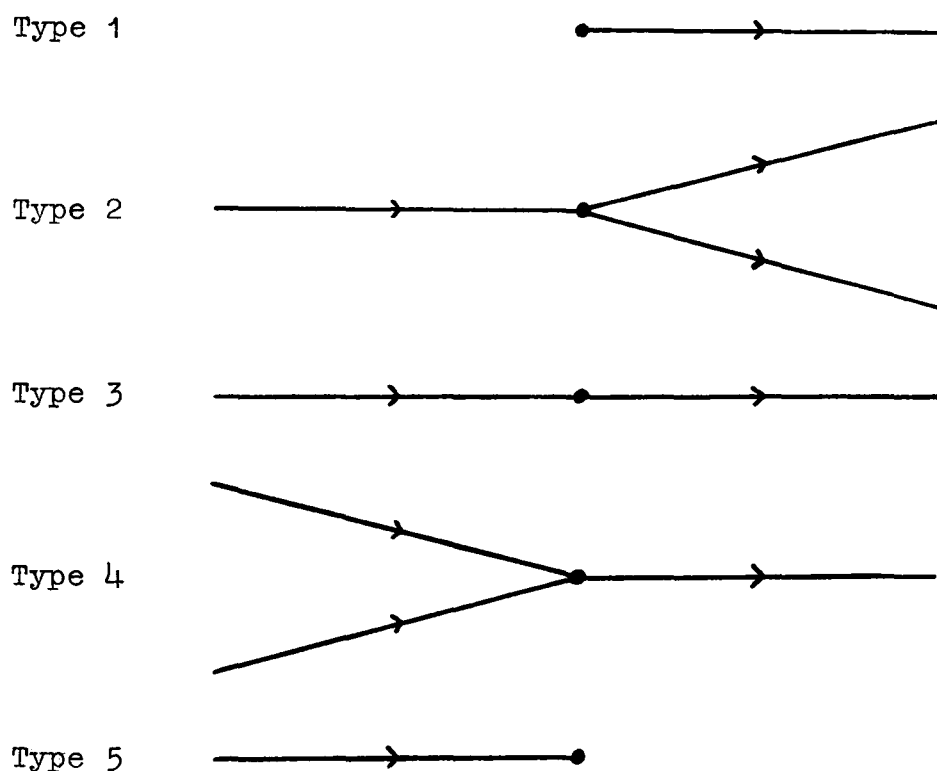


b) $\dim K \geq 2$. If $V \times \mathbb{R}$ has two components then each component meets $A(U)$ in a connected set. Thus the way in which each component is attached to $A(U)$ is determined by a single transit function for each component. Since each component of $V \times \mathbb{R}$ is itself a product, the result is independent of the isotopy type of the attaching map.

Suppose that $V \times \mathbb{R}$ is connected. If the intersection of $V \times \mathbb{R}$ with $A(U)$ has two components then both must have the same isotopy type of attaching map, as in case (a). If the intersection is connected we have only one isotopy type to consider, and again since $V \times \mathbb{R}$ is a product the result is independent of the isotopy type of the transit function. ||

Thus if H is a stable Hamiltonian function with C.S.I. (X, D) , and K_{β_i} is a manifold w.o.s. of the foliation, then K_{β_i} uniquely determines X_{β_i} . For points and manifolds it is also true that K_{β_i} uniquely determines X_{β_i} , Hence when H is a stable Hamiltonian function we can replace $X = \{X_{\beta_i}\}$ by $K = \{K_{\beta_i}\}$ in the C.S.I. We call (K, D) a simplified complete set of invariants (S.C.S.I.) .

Next we investigate the graph of a stable Hamiltonian function H on a manifold w.b. M^n . We show that the graph has a characteristic form, viz. in addition to the properties of the previous sections, all its vertices belong to one of the five types:



If we adopt the convention, mentioned above, that when a function H induces a Hamiltonian foliation F then we orient the graph in the direction in which $h = H \circ q^{-1}$ is increasing, then we have the following results:

Theorem 2

a) If v is a vertex of Γ of type 1, then $q^{-1}(v)$ is a boundary component of M on which H is a local minimum, or $q^{-1}(v)$ is a critical point of H of Morse index 0.

b) If v is a vertex of Γ of type 2, then $q^{-1}(v)$ is a manifold w.o.s. and the singularity is a critical point of H of Morse index 1.

c) If v is a vertex of Γ of type 3, then $q^{-1}(v)$ is a manifold w.o.s. and the singularity is a critical point of H of Morse index λ , where $1 \leq \lambda \leq n-1$.

d) If v is a vertex of Γ of type 4, then $q^{-1}(v)$ is a manifold w.o.s. and the singularity is a critical point of H of Morse index $n-1$.

e) If v is a vertex of Γ of type 5, then $q^{-1}(v)$ is a boundary component of M on which H is a local maximum, or $q^{-1}(v)$ is a critical point of H of Morse index n .

f) Every vertex v of Γ is one of the above types.

Proof The proof of the theorem follows very easily from results we have already. The association of vertex type with Morse index comes from looking at the energy surfaces in a neighbourhood of the critical

point involved. We leave the details to the reader. ||

Proposition 3 Let Γ be a graph each of whose vertices belongs to one of the above types. Let V_i denote the number of vertices of Γ of type i . Then

$$V_1 + V_2 = V_4 + V_5 .$$

Proof By induction on the number of vertices.

Any graph of the above type must have at least two vertices, and the only graph with two vertices is



and for this graph the result holds. Now suppose that the result holds for all graphs with $< n$ vertices, and let Γ be a graph with n vertices.

Γ contains a vertex v of type 1. Let v' be the vertex of Γ adjacent to v .

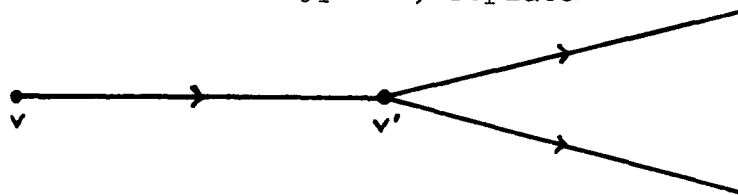
a) If v' is of type 5, remove both v and v' and the edge joining them. We now have a graph with $n - 2$ edges for which the result holds. Hence the result for Γ .

b) If v' is of type 4, remove v and the edge joining it to v' . The new graph has $n - 1$ vertices, one less vertex of type 1, one less vertex of type 4, and one more vertex of type 3. The result

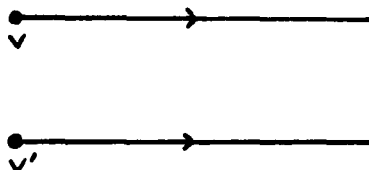
holds for the new graph and hence for Γ .

c) If v' is of type 3, remove v and the edge joining it to v' . The result holds for the new graph which has $n - 1$ vertices and the same numbers of vertices of types 1, 2, 4, 5, as Γ .

d) If v' is of type 2, replace



by



The new graph has n vertices, one less of type 2, and one more of type 1. Hence if the proposition were true for Γ then it is true for the new graph, and conversely. So we repeat the above procedure on the new graph. Γ must have $< \frac{1}{2}n$ vertices of type 2. Hence in $< \frac{1}{2}n$ steps we must arrive at a graph in which v is not adjacent to a type 2 vertex, and then the result follows by the induction argument. \parallel

3.8 \mathbb{E} -stability of a Stable Hamiltonian Function.

We give below a proof of the \mathbb{E} -stability of a stable Hamiltonian function. The stability is in fact

implied for almost all stable Hamiltonian functions by the equivalent stability proof in the theory of singularities; the equivalence relation used there is slightly stronger than our \mathbb{E} -equivalence. However we give a proof of \mathbb{E} -stability below for completeness and because our singular bundle techniques give a nice geometrical proof.

We first look at the stability problem on a trivial singular bundle. We show that the standard function on the trivial singular bundle E of a manifold w.o.s. K is \mathbb{E} -stable. This is a stronger result than will be needed immediately but will be useful later.

Theorem 1 Let K be a manifold w.o.s., (π, E, K) the trivial singular bundle over K , and H the canonical function on E . Then H is \mathbb{E} -stable.

Proof Let H_1 be a function on E C^2 close to H . We must show that if H_1 is a sufficiently C^2 -small perturbation of H then H and H_1 are \mathbb{E} -equivalent.

Let $m \in K \subset E$ be the critical point of H . Then there exists a coordinate chart (U, ϕ) on a neighbourhood U of m in E w.r.t. which $m = \phi^{-1}(0)$ and H has the canonical form

$H(x_1, \dots, x_n) = x_1^2 + \dots + x_\lambda^2 - x_{\lambda+1}^2 - \dots - x_n^2$
 in the local coordinates. We may suppose that $\varphi(U)$
 is the open ball $\{x : \|x\| < k\} \subset \mathbb{R}^n$ for some
 $k > 0$, and $n = \dim E$. Let $V = \varphi^{-1}(\{x : \|x\| < \frac{1}{2}k\})$.
 It is a standard result that if H_1 is sufficiently
 C^2 close to H then H_1 has a unique critical point
 m_1 near m and of the same Morse index as m . Thus
 we may suppose that $m_1 \in V$.

We define a function H' as follows: Let α be
 a smooth function on E such that $\alpha \equiv 1$ on V ,
 $\alpha \equiv 0$ in a neighbourhood of $E - U$, and $\alpha \in (0, 1)$
 elsewhere. Define

$$H'(p) = \alpha(p) \cdot H_1(p) + (1 - \alpha(p))H(p).$$

Then we assert that H' also has a unique critical
 point m_1 . For certainly H' has no other critical
 point inside V or outside U . In $U - V$ we have

$$\begin{aligned}
 |dH'(x)| &= |d\alpha(x) \cdot H_1(x) + \alpha(x) \cdot dH_1(x) + \\
 &\quad (1 - \alpha(x))dH(x) - d\alpha(x) \cdot H(x)| \\
 &= |dH(x) + d\alpha(x)(H_1(x) - H(x)) \\
 &\quad + \alpha(x)(dH_1(x) - dH(x))| \\
 &\geq |dH(x)| - |d\alpha(x)| \cdot |H_1(x) - H(x)| \\
 &\quad - |\alpha(x)| \cdot |dH_1(x) - dH(x)|.
 \end{aligned}$$

Now $|\alpha|$ and $|d\alpha|$ are bounded and $|dH(x)|$ is
 bounded below on $U - V$, so taking H_1 sufficiently
 C^2 close to H we have $|dH'(x)| > 0$ on $U - V$.

We can find a coordinate chart (W, ψ) about m_1

w.r.t. which H' has the standard form.

First we show that we may suppose that $W = U$. We begin by looking at H' in the local coordinates given by (U, ϕ) . We have

$$H'(x) = H'(0) + \sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \mu_i x_i^2 + \sum_{i,j=1}^n \nu_{ij} x_i x_j + F(x)$$

where $H(x) = \sum_{i=1}^n \mu_i x_i^2$, λ_i, ν_{ij} are small of order ε , and $F(0) = 0$, $DF(0) = 0$, and $D^2F(0) = 0$.

Now by a small (of order ε) affine linear coordinate change, isotopic to 1, we can reduce H' to the form

$$H'(x) = \sum_{i=1}^n \mu_i x_i^2 + F'(x),$$

where $F'(0) = 0$, $DF'(0) = 0$, $D^2F'(0) = 0$, and this coordinate change is valid throughout U . Looking at the proof of Morse Lemma ([1] or [2]) we see that it remains to go through the non-linear orthogonalisation process. But in order to show that the orthogonalisation is valid everywhere in U it is certainly sufficient to show (using Wall's notation in [1]) that $g_{ii}(x) = \mu_i + O(\varepsilon)$ and $g_{ij}(x) = O(\varepsilon)$ ($i \neq j$) throughout U . For then the first step in the process is valid everywhere, and in succeeding steps we have $g'_{ij}(x) = g_{ij}(x) + O(\varepsilon^2)$.

But

$$g_{ii}(x) = \int_0^1 \frac{\partial}{\partial x^i} \left(\int_0^1 \frac{\partial H'(sx)}{\partial x^i} ds \right) (tx) dt$$

$$= \mu_i + O(\varepsilon) .$$

Similarly $g_{ij}(x) = O(\varepsilon)$, $i \neq j$. Thus we have determined a coordinate chart (W, ψ) w.r.t. which H' has the canonical form and $W = U$. Note that since this last coordinate change is isotopic to 1 and since H and H' agree on a neighbourhood of $E - U$ we can find a neighbourhood N of $\{ \|x\| = k \}$ in $\{ \|x\| \leq k \} \subset \mathbb{R}^n$ such that the map $\psi \circ \phi^{-1}$ defined on N is isotopic to 1 and preserves the foliation given by $\sum_{i=1}^n \mu_i x_i^2$ on N . Hence we can extend it to a foliation preserving mapping of $\{ \|x\| < k \}$ onto itself. Call this mapping χ .

We can at last prove the first step in the result, viz. that H and H' are \mathbb{R} -equivalent. We define the foliation preserving diffeomorphism by

$$\begin{aligned} f(x) &= x , \text{ outside } U , \\ f(x) &= \psi^{-1} \circ \chi \circ \phi(x) , \text{ on } U . \end{aligned}$$

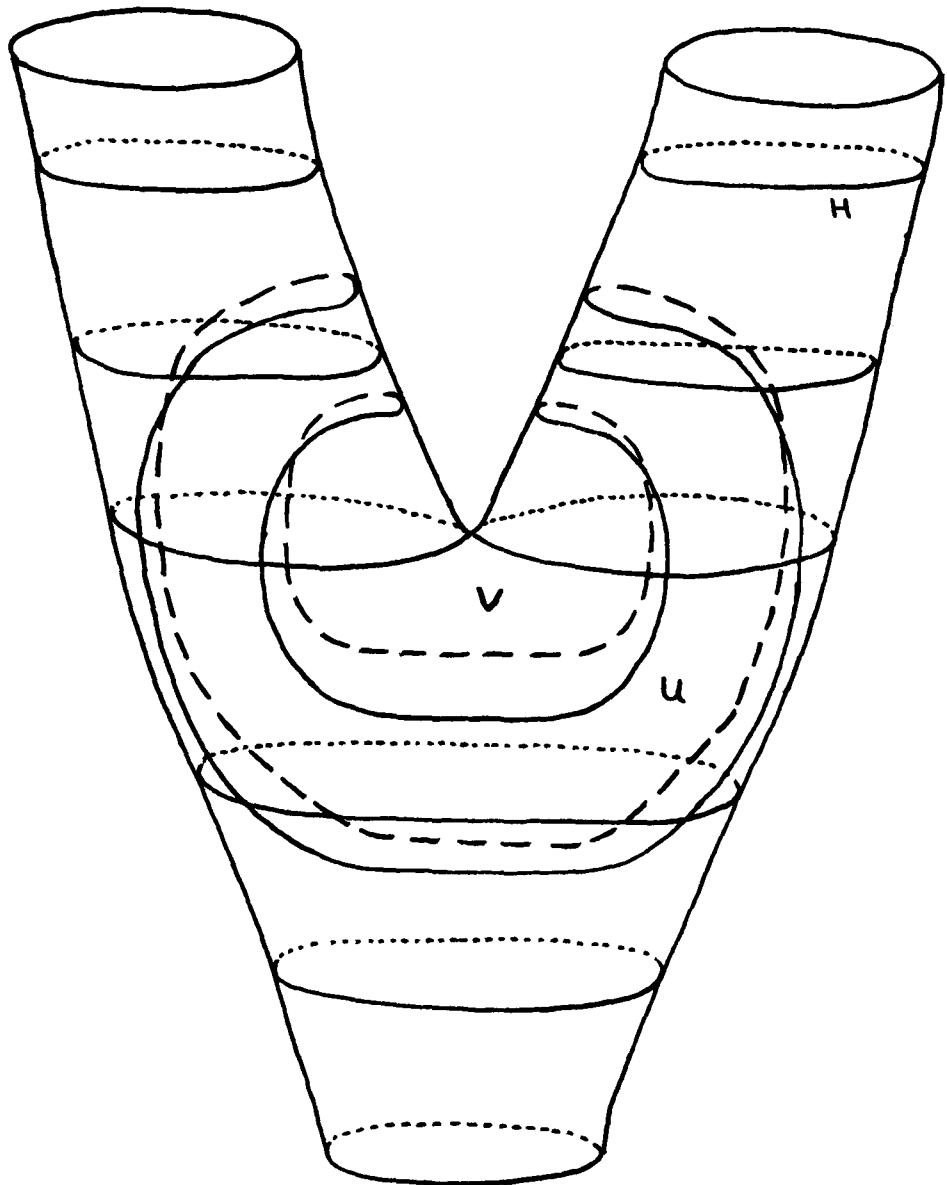
Now let X be a closed tubular neighbourhood of K in E whose boundary is a union of energy surfaces of H , and let Y be a closed tubular neighbourhood of $H_1^{-1}(0)$ in E whose boundary is a union of energy surfaces of H_1 , We may suppose that $Y \subset \overset{\circ}{X}$, and we can find a smooth function α so that $\alpha \equiv 1$ on Y , $\alpha \equiv 0$ outside X , and $\alpha \in (0,1)$ elsewhere. Define a function H'' on E by

$$H''(x) = \alpha(x) \cdot H_1(x) + (1 - \alpha(x))H(x) .$$

Precisely as before we show that H'' has a unique critical point at m_1 .

Choose a Riemannian metric on E . Then the inverse gradient of H' gives a fibration of E w.r.t. which H' is the canonical function. Now by a careful choice of X , Y , and α , we can ensure that there is a neighbourhood of the fibre over m_1 on which H' and H'' agree . We can construct an \mathbb{R} -equivalence of H' with H'' : Using the same Riemannian metric we have an inverse gradient flow for H'' which gives a fibration of E . We define an equivalence diffeomorphism ϕ as follows

- a) Define ϕ to be the identity on the neighbourhood of the fibre over m_1 on which H' and H'' agree.
- b) We can write $\partial X = \partial X_+ \cup \partial X_-$ where H is positive on ∂X_+ and negative on ∂X_- . At least one of ∂X_+ , ∂X_- is connected. Suppose ∂X_+ is connected. On the component of $\overline{E - X}$ which contains ∂X_+ we have $H' = H''$. Define ϕ to be the identity on this set also.
- c) Suppose x is not in the domain of ϕ as defined so far. Let ψ and χ be the inverse gradient flows for H' and H'' resp. x is not in the fibre over m_1 , so $\psi_{t_0}(x) \in \partial X_+$ for some t_0 . Define ϕ by



We define the \mathbb{E} -equivalence between H and H_1 by means of two intermediate functions H' and H'' . Define H' to be a function equal to H_1 on V and equal to H outside U .

Figure 4

$$\varphi(x) = \chi_{-t_0}(\psi_{t_0}(x)) .$$

Clearly φ is well defined and smooth. Globally φ maps ψ onto χ fibrewise and preserving the fibre parameter, and hence φ maps H' onto H'' preserving the foliation. Hence φ is an \mathbb{E} -equivalence of H' with H'' .

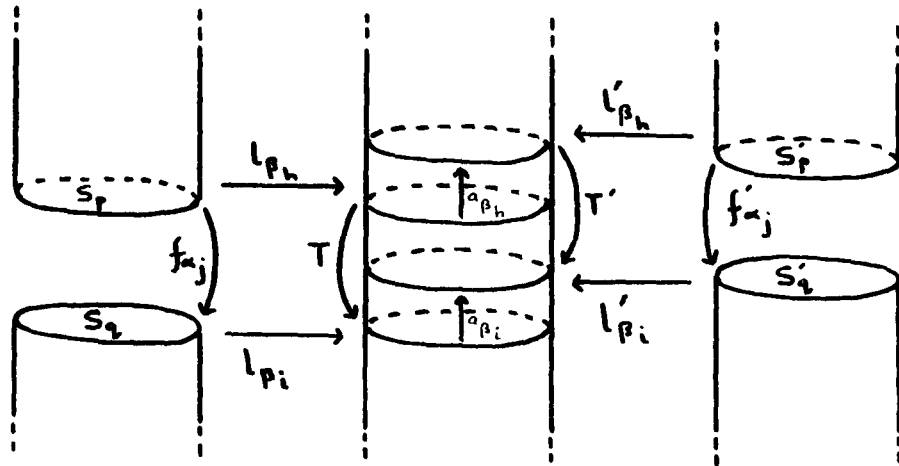
Finally H'' and H_1 are \mathbb{E} -equivalent by Proposition 3.5.2 since they agree on a neighbourhood of $H_1^{-1}(0) \cong K$. ||

The global stability theorem is now fairly easy. Effectively it remains to show that the isotopy classes of the maps f_{α_j} are unchanged.

Theorem 2 Let H be a stable Hamiltonian function on a manifold w.b. M . Then there exists a neighbourhood θ of H in $\mathcal{H}(M)$ with the C^2 topology such that $K \in \theta$ implies that H and K are \mathbb{E} -equivalent.

Proof Let H have C.S.I. (X, D) and K have C.S.I. (X', D') . Then by Theorem 1 , if K is sufficiently C^2 close to H then we have a bijection $X \longrightarrow X' :$
 $X_{\beta_i} \longrightarrow X'_{\beta_i}$, a bijection $D \longrightarrow D' : [f_{\alpha_j}] \rightsquigarrow [f'_{\alpha_j}]$
 and diffeomorphisms $X_{\beta_i} \longrightarrow X'_{\beta_i}$. For we have an embedding of each X_{β_i} in M , and so we can pull

back onto X_{β_i} the local K-foliation and apply the theorem. Thus for each β_i we have an \mathbb{E} -equivalence of $\text{image}(X_{\beta_i})$ with $\text{image}(X'_{\beta_i})$ in M . Denote this map and its restrictions to surfaces by a_{β_i} , and note that a_{β_i} is isotopic to 1 in M . Thus



using the notation of Theorem 3.5.6. Now

$$\begin{aligned} f'_{\alpha_j} &= (\iota'_{\beta_i})^{-1} \circ T' \circ \iota'_{\beta_h} \\ &\simeq (\iota'_{\beta_i})^{-1} \circ a_{\beta_i} \circ T \circ (a_{\beta_h})^{-1} \circ \iota'_{\beta_h} \end{aligned}$$

(since by the proof of Theorem 1 we may take a_{β_i} , a_{β_h} to be isotopic to translations in L_{α_j} .)

$$\begin{aligned} &= (\iota'_{\beta_i})^{-1} \circ a_{\beta_i} \circ \iota_{\beta_i} \circ f_{\alpha_j} \circ (\iota_{\beta_h})^{-1} \circ \\ &\quad (a_{\beta_h})^{-1} \circ \iota'_{\beta_h} \\ &= s_q \circ f_{\alpha_j} \circ s_p^{-1} \end{aligned}$$

since s_p , s_q , are the induced maps $(\iota'_{\beta_h})^{-1} \circ a_{\beta_h} \circ \iota_{\beta_h}$

and $(\iota'_{\beta_i})^{-1} \circ a_{\beta_i} \circ \iota_{\beta_i}$ resp. Hence H and K have the same C.S.I. and so are \mathbb{E} -equivalent. \parallel

Corollary Let X_H be a stable Hamiltonian dynamical system on a manifold w.b. M . Then there exists a neighbourhood θ of X_H in $\mathcal{X}_H(M)$ such that $X_K \in \theta$ implies that X_H and X_K are \mathbb{E} -equivalent. \parallel

Chapter 4

Hamiltonian Foliations on Compact 2-manifolds

Many of the results of chapter 3 can be strengthened considerably when we are considering a manifold w.b. of dimension 2 .

4.1 Structure of Hamiltonian Dynamical Systems

We begin by noticing that if M is an orientable 2-manifold w.b. then M is a symplectic manifold w.b. For any volume ω is a symplectic form on M . Also we note that there is essentially only one symplectic form on M , for if ω' is any other symplectic form on M then there exists a smooth non-zero function f on M such that $\omega(x) = f(x) \cdot \omega'(x)$ for all $x \in M$.

Theorem 1 Let $X_H \in \mathcal{X}_N(M)$, $\dim M = 2$. Then

- a) every critical point of X_H is either a centre or a saddle point,
- b) every point $x \in M$ is either

- 1) a fixed point of X_H ,
 or 2) on the stable manifold of a saddle point,
 or 3) on a periodic orbit.

Proof Notice that each component of an energy surface of H is in fact an orbit of X_H . This follows from the observation that each energy surface is a one dimensional manifold and contains no critical points.

(a) then follows from the Morse Lemma for the canonical form of H near a non-degenerate critical point. To prove (b) we note that a regular energy surface of H is a closed 1-manifold, i.e. S^1 , and if x is on a singular energy surface which does not intersect ∂M then the closed energy surface is a 1-manifold w.s. whose singularities are critical points of X_H . The result then follows from our initial observation. ||

Corollary Every stable manifold of X_H is also an unstable manifold. ||

This is in contrast to the corresponding result for Morse-Smale systems on M^2 , for which stable and unstable manifolds intersect transversely.

Theorem 2 Let $X_H \in \mathcal{X}_N(M)$, $\dim M = 2$. Let $K =$

$\{x : x \text{ is a critical point of } X_H \text{ or on a stable manifold of a critical point}\}$, and $L = \{x : x \text{ is on a closed orbit of } X_H\} = M - K$. Then L is a finite disjoint union of open annuli $\cong S^1 \times \overset{\circ}{I}$ and L is dense in M .

Proof Theorems 3.1.2 and 4.1.1 . ||

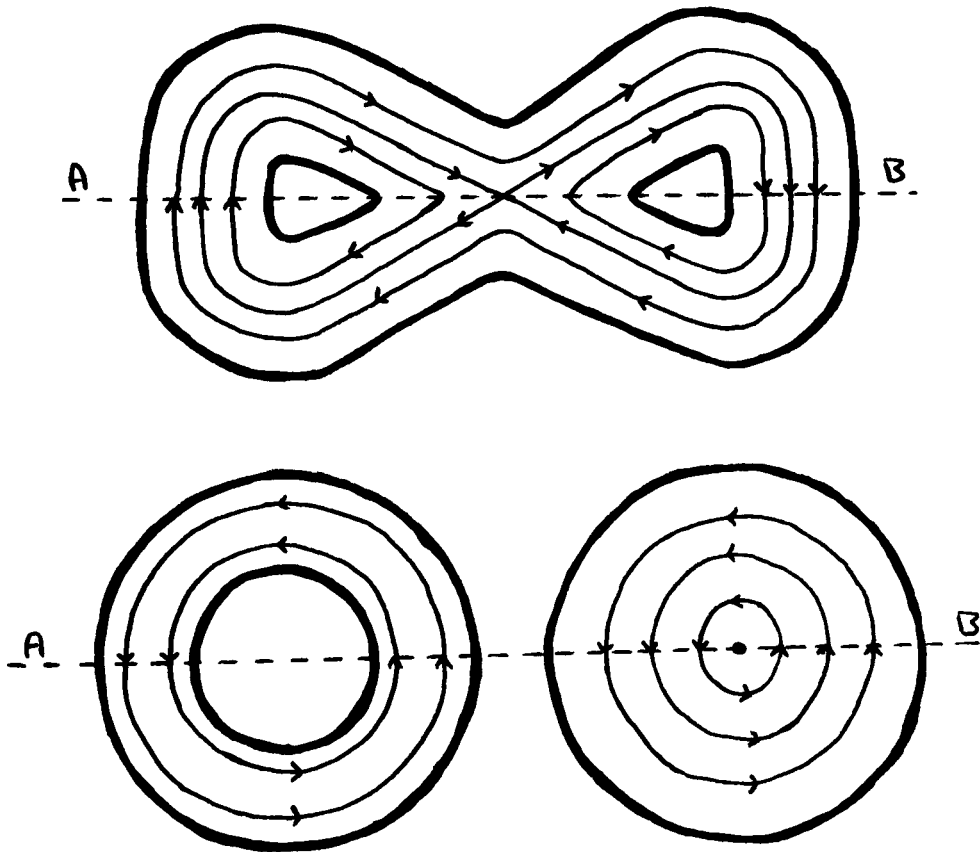
4.2 Stable Hamiltonian Systems

We will prove the result that if M is a symplectic 2-manifold w.b. and $X_H, X_K \in \mathcal{X}_S(M)$, then X_H and X_K are \mathbb{E} -equivalent if and only if X_H and X_K are \mathbb{H} -equivalent. The method is as follows: First, \mathbb{H} -equivalence implies \mathbb{E} -equivalence. For it is clear that the results of Theorem 4.1.2 and the techniques of chapter 3 enable us to smooth an \mathbb{H} -equivalence homeomorphism to an \mathbb{E} -equivalence diffeomorphism. Suppose that X_H and X_K are \mathbb{E} -equivalent. Then we have a diffeomorphism $\phi : M \longrightarrow M$ which maps energy surfaces of H onto those of K . Therefore ϕ maps orbits onto orbits, but may not preserve the sense of an orbit. Hence it is sufficient to show

Lemma 1 Let X_H be a stable Hamiltonian dynamical system on M^2 . Then if M is symplectic (i.e. orientable) there exists a homeomorphism $h : M \longrightarrow M$ which

maps every orbit onto itself but reverses the sense of the orbit.

Proof It is sufficient to show that the homeomorphism exists in each of the three types of canonical neighbourhood and is consistent on their intersections. Since M is orientable and H is stable the three possible canonical neighbourhoods are



and the required map is given by reflection in the

dotted line AB . We can arrange these reflections to be consistent on the intersection of two canonical neighbourhoods. ||

Now if the \mathbb{E} -equivalence ϕ reverses sense at one point in M , then it reverses sense at every point of M . (Look at ϕ on the canonical neighbourhoods.) Hence either ϕ or $\phi \circ h$ is an \mathbb{H} -equivalence. We have proved

Theorem 2 Suppose that M is a symplectic 2-manifold w.b. and $X_H, X_K \in \mathcal{X}_S(M)$. Then X_H and X_K are \mathbb{E} -equivalent if and only if X_H and X_K are \mathbb{H} -equivalent. ||

Corollary If $X_H \in \mathcal{X}_S(M)$, $\dim M = 2$, then X_H and X_{-H} are \mathbb{H} -equivalent. ||

4.3 Homology Properties

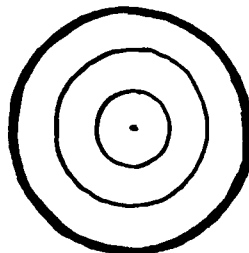
In the next two sections we show that for 2-manifolds we can characterise the stable Hamiltonian functions up to \mathbb{E} -equivalence (and therefore up to \mathbb{H} -equivalence when the manifold is orientable) by their graphs and additional information depending on the manifold. Thus we can dispense with the C.S.I.

In this section we investigate the relationship

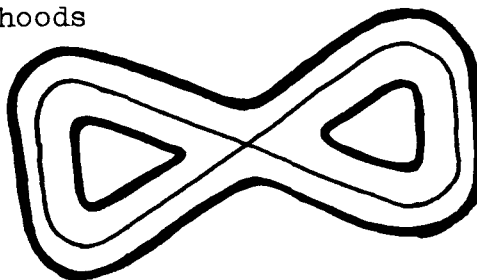
between the homology of a manifold w.b. and the homology of the graph of a stable Hamiltonian function on the manifold w.b. For simplicity we will deal only with closed two-manifolds and indicate at the end of the section how to deal with 2-manifolds w.b.

Theorem 1 Suppose that M is an orientable 2-manifold and $H \in \mathcal{H}_S(M)$. Let $\Gamma = \Gamma(H)$. Then

- a) Γ contains no vertices of type 3 ,
- b) vertices of Γ of types 1 and 5 correspond to canonical neighbourhoods of critical points



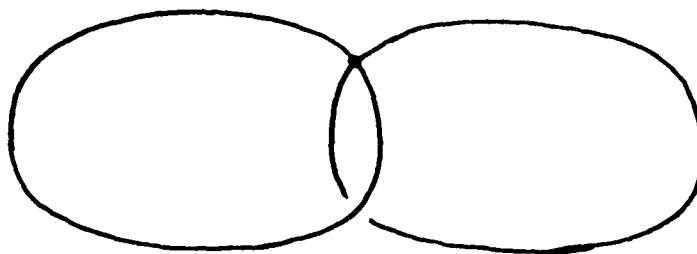
- c) vertices of Γ of types 2 and 4 correspond to canonical neighbourhoods



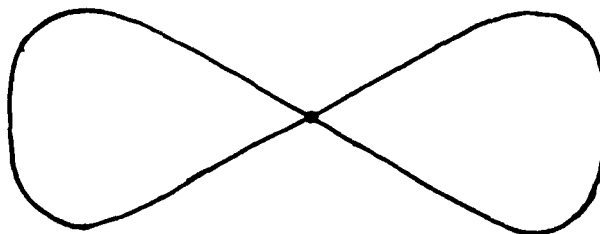
- d) $H_1(M) \cong H_1(\Gamma) \oplus H_1(\Gamma)$

Proof We begin by enumerating the possible sets K_{β_i} in the foliation induced by H . Since $\partial M = \emptyset$, no

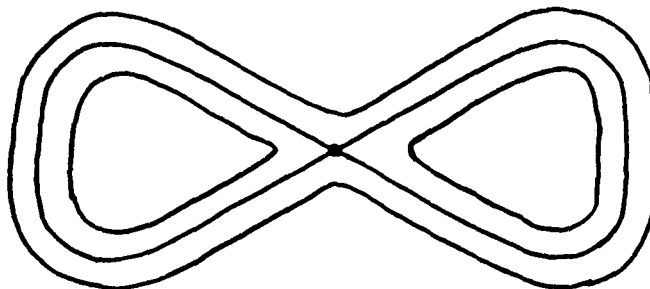
boundary components S^1 are possible. Since



is not an orientable manifold w.o.s., no K_{β_i} can have this form. The remaining possibilities are isolated critical points and manifolds w.o.s. isomorphic to K :



Clearly a canonical neighbourhood of an isolated critical point gives rise to a vertex of type 1 or 5 in Γ , and a canonical neighbourhood of K is isomorphic to



and so gives rise to a vertex of type 2 or 4 . This proves (a) , (b) , and (c) .

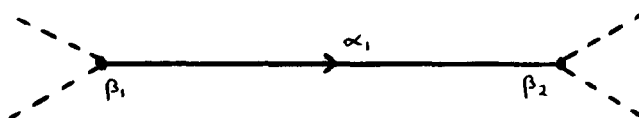
d) Since $H_1(M)$ and $H_1(\Gamma)$ are free groups on the integers it is sufficient to show that

$$\dim H_1(\Gamma) = \frac{1}{2} \dim H_1(M) .$$

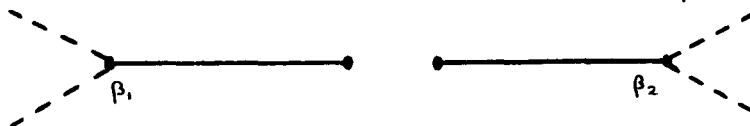
We will do this by a series of lemmas:

Lemma 2 If $M \cong S^2$ then Γ contains no circuits.

Proof Suppose that there exists a circuit and let α_1 be an edge in the circuit



Choose a point x in this edge and cut S^2 along $q^{-1}(x)$. Then the new manifold w.b. has two components and graph Γ' equal to Γ with edge α_1 replaced by



Thus the graph Γ' is connected but the manifold w.b. is not. Contradiction. ||

Lemma 3 $\dim H_1(\Gamma) \leq \frac{1}{2} \dim H_1(M)$.

Proof M is an orientable 2-manifold and so is diffeomorphic to a sphere with n handles attached. We prove the lemma by induction on the number of handles.

The result is true for 0 handles by the above lemma. Suppose that it is true when M' is a sphere with $n-1$ handles, and let M be a sphere with n handles. Suppose that there exists a circuit in Γ . Choose an edge α_1 in this circuit and a point $x \in \alpha_1$. Cut M along $q^{-1}(x)$. Then the new manifold w.b. has two boundary components, each diffeomorphic to S^1 . To each component we glue a disc with canonical foliation, and thus obtain a closed manifold M' and a stable Hamiltonian function H' on it. (H' is defined up to \mathbb{E} -equivalence by the foliation on M' .) Now M' is a sphere with $n-1$ handles, and the graph Γ' of H' has one less circuit. The result follows by induction. ||

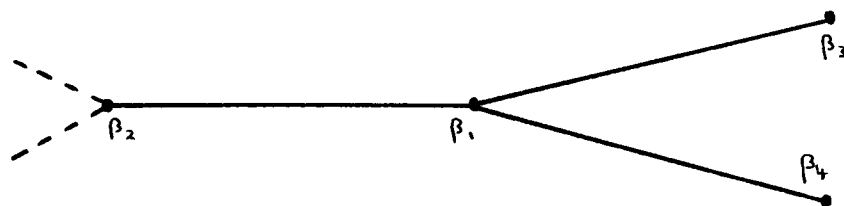
Lemma 4 If Γ is simply connected then $M \cong S^2$.

Proof Induction on the number of edges of Γ . If Γ has one edge then the graph is



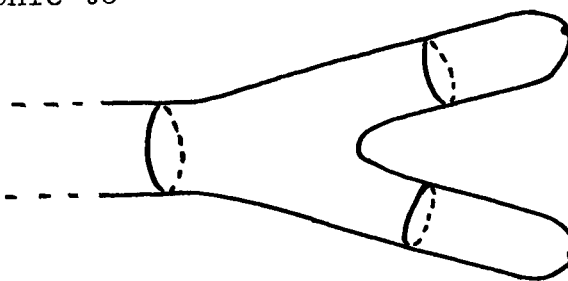
and the result is immediate. Γ cannot have two edges since no vertices of type 3 are allowed. Suppose that the result is true for graphs Γ' with $\leq n$ edges, $n \geq 2$, and let Γ have $n+1$ edges. Then we can find a vertex β_1 of Γ of type 2 or 4 with two adjacent

vertices of type 1 or 5 . i.e.

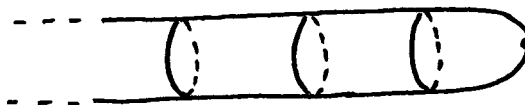


For choose a vertex β of Γ of type 2 or 4 . If two or more of its adjacent vertices are not of types 1 or 5 then choose an adjacent vertex β' of type 2 or 4 . If β' is not a vertex with the required configuration then there exists a vertex $\beta'' \neq \beta$ adjacent to β' and of type 2 or 4 . This process repeated finitely often will give us a vertex with the required configuration, for since Γ contains no circuits we can never return to a vertex already rejected, and since Γ is finite the process must end.

Now in a neighbourhood of $q^{-1}(\beta_1)$ the manifold is diffeomorphic to



Using the glueing technique of Lemma 3 we replace this by a foliation



Clearly the new manifold is diffeomorphic to the old one, but the graph Γ' of the new foliation has two less edges than Γ .

Hence the result follows from the induction hypothesis. ||

Lemma 5 $\dim H_1(\Gamma) \geq \frac{1}{2} \dim H_1(M)$.

Proof By induction on $\dim H_1(M)$. We use the same induction technique as in Lemma 3, starting the induction with the result of Lemma 4. ||

Theorem 1 now follows from Lemmas 3 and 5. ||

We next look at stable Hamiltonian functions on non-orientable 2-manifolds. Of course there are no corresponding dynamical systems on non-orientable manifolds.

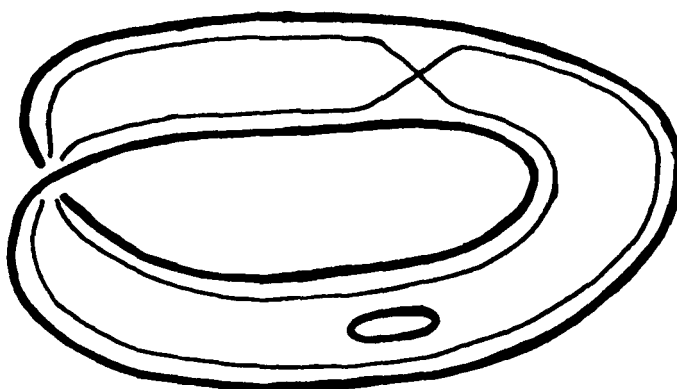
If G is a free group of the type $G = \oplus^n \mathbb{Z} = \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ (n factors), $n \geq 1$, let $(G)_2$ denote the group $\oplus^{n-1} \mathbb{Z} \oplus \mathbb{Z}_2$.

Theorem 6 Suppose that M is a non-orientable 2-manifold and H is a stable Hamiltonian function on M . Let $\Gamma = \Gamma(H)$. Then if Γ has n vertices

of type 3 ,

$$H_1(M) \cong (H_1(\Gamma) \oplus H_1(\Gamma) \oplus {}^n\mathbb{Z})_2 .$$

Also, in addition to the vertex-canonical neighbourhood correspondences of Theorem 1 , we have: Vertices of Γ of type 3 correspond to non-orientable canonical neighbourhoods



Proof The last statement is clear from the remarks in the proof of theorem 1 and an examination of the image of the above canonical neighbourhood under the map $q : M \longrightarrow \Gamma$.

To prove the result on the homology of M we again use a sequence of lemmas. We first remark that if M is a non-orientable closed 2-manifold then $H_1(M) = \oplus {}^m\mathbb{Z} \oplus \mathbb{Z}_2$, $m \geq 0$.

We will show that the result is true for the two cases

- a) $\dim H_1(\Gamma) = 0$, $n = 1$, and
- b) $\dim H_1(\Gamma) = 1$, $n = 0$.

Then we use induction on $\dim H_1(\Gamma)$ and on n . Note

that in the case $\dim H_1(\Gamma) = 0$, $n = 0$, we have already shown that $M = S^2$ which is orientable, so this case cannot arise.

Lemma 7 If $\dim H_1(\Gamma) = 0$, $n = 1$, then $M \cong P^2$.

Proof By induction on the number of edges of Γ . Since Γ contains a vertex of type 3 , Γ has at least two edges. If Γ has two edges then Γ is the graph



and so M is a Mobius strip glued to a disc, i.e. P^2 . Γ cannot have three edges since it contains precisely one vertex of type 3 . Now suppose that the result is true for all graphs Γ' with $\leq n$ edges, $n \geq 3$. We use the induction procedure of Lemma 4 starting from the vertex β of type 3 . If Γ is a graph with $n+1$ edges, $n \geq 3$, then there exists a vertex β' adjacent to β and not of type 1 or 5 . Hence β' is of type 2 or 4 , and we can use the technique of Lemma 4 . ||

Lemma 8 If $\dim H_1(\Gamma) = 1$, $n = 0$, then $M \cong K^2$, a Klein bottle.

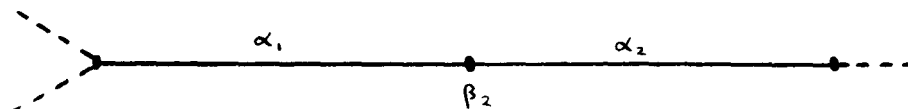
Proof We use the cutting technique of Lemma 3 . After cutting we have a new manifold M' with graph Γ' which satisfies $\dim H_1(\Gamma') = 0$, $n = 0$. Hence $M' \cong S^2$

and so M is a sphere with one handle glued non-orientably, i.e. a Klein bottle. ||

Lemma 9 If $\dim H_1(\Gamma) = 0$, and $n > 0$, then
 $H_1(M) = (\oplus^n \mathbb{Z})_2$.

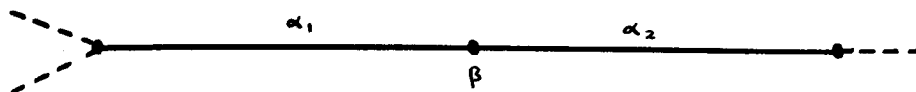
Proof The result holds for $n = 1$ by Lemma 7. Suppose that the result is true for all integers $< n$, $n \geq 2$, and suppose that Γ is a graph with n vertices of type 3. We show that we can choose one of these vertices so that all arcs in Γ connecting it to other type 3 vertices have the same first edge.

For choose β_1 of type 3. If β_1 does not have the required property choose a β_2 of type 3. The graph at β_2 looks like

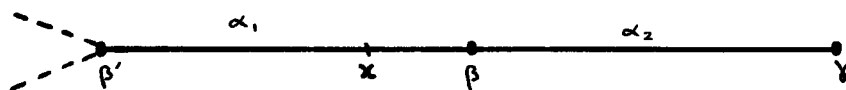


and let us suppose that the arc from β_1 to β_2 ends with edge α_1 . If β_2 is not of the required type then there exists a β_3 of type 3 such that the arc from β_2 to β_3 has first edge α_2 . Choose such a β_3 , and if β_3 is not of the required type, repeat the process. Since Γ is simply connected and finite the process must come to an end at some finite stage. Hence we can choose a β of the required type.

Suppose that we have



and all arcs joining β to other type 3 vertices have first edge α_1 . Then applying the technique of Lemma 4 to all vertices to the right of β we can find a foliation with graph Γ' on M , where $\Gamma' = \Gamma$ to the left of β and at β Γ' looks like



Cutting along $q^{-1}(x)$, $x \in \alpha_1$, the component of M containing $q^{-1}(\beta)$ is a Mobius strip. Glueing a disc to the boundary of the remainder of M we have a foliation with graph Γ'' on a manifold M'' . Γ'' is equal to Γ' to the left of β' and looks like



at β' . Γ'' has $n-1$ vertices of type 2, and M'' is homeomorphic to M with a Mobius strip removed and a disc glued in its place.

The result follows from the induction hypothesis. ||

So far we have proved that Theorem 6 holds when $\dim H_1(\Gamma) = 0$. Finally we prove

Lemma 10 Theorem 6 holds when $\dim H_1(\Gamma) = h \geq 0$.

Proof Again by induction. The result holds for $\dim H_1(\Gamma) = 0$ and $n > 0$ by Lemma 9, and for $\dim H_1(\Gamma) = 1$, $n = 0$ by Lemma 8. Hence it is sufficient to prove the result by induction on $\dim H_1(\Gamma)$ with arbitrary n . We use the cutting and glueing technique of Lemma 3. ||

Theorem 6 is now proven. ||

Let H be a stable Hamiltonian function on a 2-manifold w.b. M , and let $\Gamma = \Gamma(H)$ be its graph. Each component of ∂M is a circle S^1 , so we can form an associated manifold M' by glueing to each component of ∂M a disc. Taking the canonical foliation on each such disc we have a stable Hamiltonian function H' on M' (uniquely defined up to \mathbb{R} -equivalence), with graph Γ' say. Clearly $\Gamma' \cong \Gamma$. Thus to investigate H on M , we can form H' on M' and use the results proved above for stable Hamiltonian functions on manifolds.

The corresponding results for manifolds w.b. are:

Theorem 11 Suppose that M is an orientable 2-manifold w.b. and H is a stable Hamiltonian function on M . Let $\Gamma = \Gamma(H)$, and suppose that ∂M has $r \geq 1$ components. Then

- a) Γ contains no vertices of type 3 ,
 b) $H_1(M) \cong H_1(\Gamma) \oplus H_1(\Gamma) \oplus^{r-1} \mathbb{Z} .$ ||

Theorem 12 Suppose that M is a non-orientable 2-manifold w.b. and H is a stable Hamiltonian function on M . Let $\Gamma = \Gamma(H)$ and suppose that ∂M has $r \geq 1$ components and Γ has $n \geq 0$ vertices of type 3 .
 Then

$$H_1(M) \cong H_1(\Gamma) \oplus H_1(\Gamma) \oplus^{n+r-1} \mathbb{Z} .$$
||

4.4 Reduction of Invariants

We show that for 2-manifolds the set of invariants of a stable Hamiltonian foliation can be reduced considerably. We look first at the case when M is a closed orientable 2-manifold.

Theorem 1 Let H and H' be stable Hamiltonian functions on a closed orientable 2-manifold M . Then H and H' are \mathbb{IE} -equivalent if and only if their graphs Γ and Γ' are isomorphic.

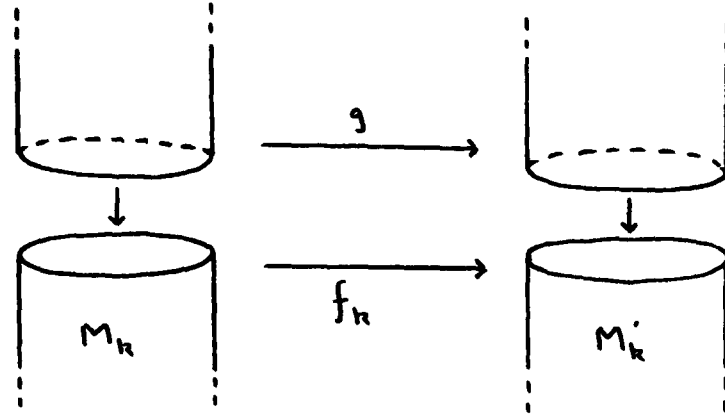
Proof Since M is a closed orientable 2-manifold we have two possible types of canonical neighbourhood for a critical point of H (see Theorem 4.3.1). Now if Γ and Γ' are isomorphic, H and H' decompose into the same set of canonical neighbourhoods, and it remains

only to show that we can choose the glueing maps correctly. We do this by an induction process, which in fact gives a direct construction of an IE-equivalence.

Order the vertices of Γ to agree with the orientation on Γ (see section 2.8). We then have an ordering of the vertices of Γ' induced by the graph isomorphism. We can construct M from the canonical neighbourhoods by taking the vertices of Γ in order and glueing the corresponding canonical neighbourhoods together in this order using the glueing maps. We do this for each foliation, and construct an IE-equivalence at each stage.

Let M_k and M'_k denote the manifolds w.b. constructed after k steps, and f_k the IE-equivalence $M_k \longrightarrow M'_k$. Since the first vertex of Γ and of Γ' is of type 1, $M_1 \cong B^2 \cong M'_1$ and we have no problem in constructing an IE-equivalence. Suppose that we have an IE-equivalence $f_k : M_k \longrightarrow M'_k$. If the $k+1$ th vertex of Γ (and therefore of Γ') is of types 1 or 5, then we have no problems in extending f_k to f_{k+1} . If the $k+1$ th vertex is of type 2, then the canonical neighbourhoods are glued to M_k and M'_k by one connected component each. Now we have $f_k : M_k \longrightarrow M'_k$ and a natural IE-equivalence of the canonical neighbourhoods. Since by Lemma 4.2.1 the canonical neighbourhood is IE-equivalent to its 'mirror image' we can choose an

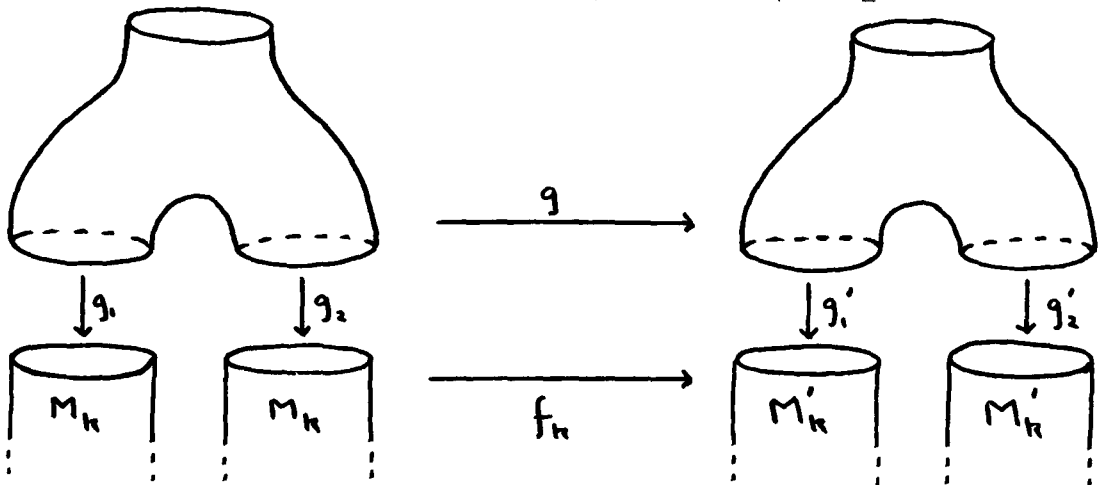
\mathbb{E} -equivalence g of the canonical neighbourhood with itself so that



commutes up to isotopy. Hence we can extend f_k to an \mathbb{E} -equivalence $f_{k+1} : M_{k+1} \longrightarrow M'_{k+1}$.

Suppose that the $k+1$ th vertex is of type 4.

Denote the glueing maps by g_1 , g_2 , g'_1 , g'_2 , as shown.



If there exists an \mathbb{E} -equivalence g of the canonical neighbourhood with itself such that $f_k \circ g_1 \simeq g'_1 \circ g$ and $f_k \circ g_2 \simeq g'_2 \circ g$ then we can extend f_k to an \mathbb{E} -equivalence f_{k+1} . If not, suppose w.l.o.g. that $f_k \circ g_1 \simeq g'_1 \circ 1$ and $f_k \circ g_2 \not\simeq g'_2 \circ 1$. Then the two surfaces

of M_k to which we are glueing cannot be in the same connected component of M_k ; for if they were, either M_k or M'_k would be non-orientable. Hence they are in different connected components of M_k and we can use the reflection map on one of these components to get a new \mathbb{E} -equivalence $M_k \longrightarrow M'_k$ which satisfies the isotopy relations and hence extends to f_{k+1} as above.

Hence we have an \mathbb{E} -equivalence at every step, and the last step gives an \mathbb{E} -equivalence of H with H' . ||

Thus

Theorem 2 Suppose that M is a closed orientable 2-manifold. Then we have a 1-1 correspondence between the set of \mathbb{H} -equivalence classes of stable Hamiltonian functions (or stable Hamiltonian dynamical systems) and the set of isomorphism classes of oriented graphs Γ such that

- a) all vertices of Γ are of types 1 , 2 , 4 , or 5 ,
- b) Γ contains no oriented circuit,
- c) $H_1(\Gamma) \oplus H_1(\Gamma) = H_1(M)$. ||

Note that the \mathbb{H} -equivalence classes of stable

Hamiltonian functions are also the \mathbb{E} -equivalence classes. To adapt the invariants for \mathbb{E} -equivalence classes to a set of invariants sufficient to classify stable functions using the equivalence relation " $H \sim H'$ " if and only if there exist diffeomorphisms $f : M \longrightarrow M$ and $k : \mathbb{R} \longrightarrow \mathbb{R}$ so that $H \circ f = k \circ H'$ " we need only add to the graph Γ an ordering of the vertices of Γ induced by the stable function, and demand that the graph isomorphism preserve this ordering. The above mentioned equivalence is commonly used in the theory of functions on manifolds.

If M is a compact orientable 2-manifold w.b. then we need some more invariants to classify the stable Hamiltonian systems up to \mathbb{H} -equivalence. Let H be a stable Hamiltonian function on M , and let $\Gamma = \Gamma(H)$. We define the set B , called the boundary set of H , to be the set of vertices $v \in \Gamma$ such that $q^{-1}(v) \subset \partial M$. Clearly B has r elements if and only if ∂M has r connected components. We call the pair (Γ, B) a boundary augmented graph, and we say that two boundary augmented graphs (Γ, B) and (Γ', B') are isomorphic if there exists an isomorphism $\Gamma \longrightarrow \Gamma'$ inducing a bijection of B onto B' .

Theorem 3 Let H and H' be stable Hamiltonian

functions on a compact orientable 2-manifold $\check{M}_{w.b.}$. Then H and H' are \mathbb{E} -equivalent if and only if their boundary augmented graphs (Γ, B) and (Γ', B') are isomorphic.

Proof We use a constructive proof as in Theorem 1. The details are almost identical. ||

Theorem 4 Suppose that M is a compact orientable 2-manifold w.b. Then we have a 1-1 correspondence between the set of \mathbb{H} -equivalence classes of stable Hamiltonian functions (or stable Hamiltonian dynamical systems) and the set of isomorphism classes of boundary augmented oriented graphs (Γ, B) such that

- a) all vertices of Γ are of types 1, 2, 4, or 5,
- b) Γ contains no oriented circuit,
- c) $H_1(\Gamma) \oplus H_1(\Gamma) \oplus^{r-1} \mathbb{Z} = H_1(M)$, where r is the number of connected components of ∂M . ||

Note that theorem 4 holds only when $r \neq 0$ and $B \neq \emptyset$.

We now look at the case when M is a closed non-orientable 2-manifold. In this case we will be classifying Hamiltonian functions only, since there are no Hamiltonian dynamical systems on non-orientable manifolds. Again

the graph alone is not sufficient to classify stable Hamiltonian systems up to \mathbb{E} -equivalence.

Let H be a stable Hamiltonian function on M and $\Gamma = \Gamma(H)$. We define an index for each edge of Γ as follows: Take a C.S.I. for H and on each orientable canonical neighbourhood define a leaf orientation. Corresponding to each edge α_j of Γ we have a glueing map f_{α_j} of the surfaces of the canonical neighbourhoods. We associate to α_j the index 0 if f_{α_j} is orientation preserving, and 1 otherwise (including the case when orientability of f_{α_j} is not defined). If Γ has p edges then we have associated to Γ a p -tuple $(e_1, \dots, e_p) = \underline{e}$ (where e_j is the index of α_j) which we call the index of H . This index will describe the glueing maps f_{α_j} , giving us sufficient information to construct the foliation from a knowledge of the graph and index of H .

We define an equivalence on the set of indices of H . If $\alpha_i, \alpha_j, \alpha_k$, are edges adjacent to a common vertex β of type 2 or 4, let \underline{e}_{ijk} be the p -tuple with 1 in the i , j , and k th places and 0 elsewhere. If \underline{e} is an index of H then \underline{e} and $\underline{e} + \underline{e}_{ijk}$ are equivalent, where $+$ denotes coordinatewise addition modulo 2. If α_i is adjacent to a vertex β of type 1, 3, or 5, let \underline{e}_i be the p -tuple with 1 in the

i th place and 0 elsewhere. If \underline{e} is an index of H then \underline{e} and $\underline{e} + \underline{e}_i$ are equivalent. Extend to an equivalence relation on the set of indices of H :

$\underline{e} \sim \underline{f}$ if $\underline{f} = \underline{e} + \underline{e}_{i_1} + \dots + \underline{e}_{i_m} + \underline{e}_{j_1 k_1 l_1} + \dots + \underline{e}_{j_n k_n l_n}$ for some $i_1, \dots, j_n, k_n, l_n$. Then $\underline{e} \sim \underline{f}$ if and only if $\underline{e} + \underline{f} \sim \underline{0}$ where $\underline{0}$ is the zero p -tuple.

We define an index augmented graph of H to be a pair (Γ, \underline{e}) where Γ is the graph of H and \underline{e} is an index of H . We say that two index augmented graphs (Γ, \underline{e}) and $(\Gamma', \underline{e}')$ are isomorphic if there exists an isomorphism $g : \Gamma \longrightarrow \Gamma'$ such that $\underline{e}' + g_*(\underline{e}) \sim \underline{0}$, where g_* is the induced map $g_*(\underline{e}) = (e_{g(1)}, \dots, e_{g(p)})$, $\alpha_{g(j)} = g(\alpha_j)$. In particular, if $\underline{e} \sim \underline{f}$ then (Γ, \underline{e}) is isomorphic to (Γ, \underline{f}) .

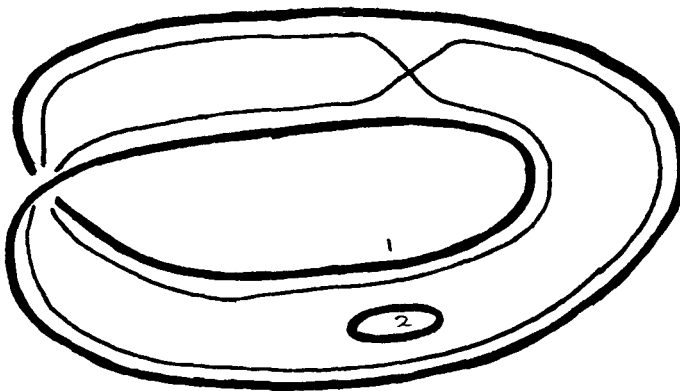
We assert that the isomorphism classes of index augmented graphs classify the \mathbb{E} -stability classes of stable Hamiltonian functions on closed non-orientable manifolds M . The indeterminacy of the index of a function results from the fact that there are many possible ways of choosing leaf orientations from which we define the index. Thus changing an index by addition of a p -tuple \underline{e}_{ijk} corresponds to choosing a different leaf orientation on the canonical neighbourhood of type 2 or 4 which intersects each of $L_{\alpha_i}, L_{\alpha_j}, L_{\alpha_k}$. If we can change an index by addition of a p -tuple \underline{e}_i

then the resulting manifold must be independent of the glueing map f_{α_i} .

Proposition 5 Let H be a stable Hamiltonian function on a closed non-orientable 2-manifold M . If (Γ, \underline{e}) is an index augmented graph for H , then so is (Γ, \underline{e}') whenever $\underline{e} \sim \underline{e}'$.

Proof It is sufficient to prove that if (Γ, \underline{e}) is an index augmented graph for H , then so are $(\Gamma, \underline{e} + \underline{e}_i)$ and $(\Gamma, \underline{e} + \underline{e}_{ijk})$ where \underline{e}_i and \underline{e}_{ijk} are defined in the circumstances described above.

In the first case the equivalence $\underline{e} \sim \underline{e} + \underline{e}_i$ means that we can choose the index e_i associated to the edge α_i at will, and hence it is sufficient to prove that the resulting manifold is independent of the isotopy class of the glueing map. When α_i is adjacent to a vertex β of type 1 or 5 this is clear, for the canonical neighbourhood of $q^{-1}(\beta)$ is a disc. If β is a vertex of type 3, then $q^{-1}(\beta)$ has canonical neighbourhood X :



and we see that there exists an \mathbb{E} -equivalence of X with itself which maps boundary 1 onto itself by a half rotation and boundary 2 onto itself by a reflection. Hence if we have a canonical neighbourhood Y glued to boundary 2 by a map f_{α_i} , the result is \mathbb{E} -equivalent to glueing by f_{α_i} followed by a reflection, which is not isotopic to f_{α_i} .

The second case is also elementary. Let β_h be a vertex adjacent to each of $\alpha_i, \alpha_j, \alpha_k$. Then $\underline{e} + \underline{e}_{ijk}$ is the index we get when we choose a different leaf orientation on X_{β_h} . ||

Theorem 6 Let M be a closed non-orientable 2-manifold, and H, H' two stable Hamiltonian functions on M with index augmented graphs (Γ, \underline{e}) and (Γ, \underline{e}') resp. Then H and H' are \mathbb{E} -equivalent if and only if (Γ, \underline{e}) and (Γ, \underline{e}') are isomorphic.

Proof By construction as in Theorem 1. We choose leaf orientations on each orientable canonical neighbourhood and then use the index to determine whether or not to glue orientably. We begin by choosing an isomorphism $g : (\Gamma, \underline{e}) \longrightarrow (\Gamma, \underline{e}')$ and note that we may suppose that $\underline{e}' = g_*(\underline{e})$. Now use the construction technique of Theorem 1. ||

Theorem 7 Suppose that M is a closed non-orientable 2-manifold. Then we have a 1-1 correspondence between the set of \mathbb{E} -equivalence classes of stable Hamiltonian functions on M and the set of isomorphism classes of index augmented oriented graphs (Γ, \underline{e}) such that

- a) all vertices of Γ are of types 1, 2, 3, 4, or 5,
- b) Γ contains no oriented circuit,
- c) $(H_1(\Gamma) \oplus H_1(\Gamma) \oplus \mathbb{Z}^n)_2 = H_1(M)$, where n is the number of vertices of Γ of type 3,
- d) if $n = 0$, then $\underline{e} \neq \underline{0}$.

Proof If $n = 0$ and $\underline{e} \neq \underline{0}$ then the resulting manifold is orientable, and so we must exclude this case. Otherwise the result is a corollary of previous theorems. ||

Again we deal with compact non-orientable 2-manifolds w.b. by adding to the invariants a boundary set B . The triple $(\Gamma, \underline{e}, B)$ then classifies the equivalence classes as before.

This completes the classification of stable Hamiltonian functions on compact 2-manifolds w.b.

Chapter 5

Hamiltonian Foliations on non-compact Manifolds

In this chapter we extend the definitions and results developed in preceeding chapters to the case of a non-compact manifold M . For simplicity we assume that $\partial M = \emptyset$, and note that we can in fact use the techniques of this chapter and of chapter 3 to apply the results below to a non-compact manifold w.b. M when $\partial M \neq \emptyset$ and each connected component of ∂M is compact.

If any previously defined terms are introduced below without being explicitly redefined, we will assume that the previous definition is still in force.

Our strategy in this chapter will be to restrict attention to those Hamiltonian dynamical systems and functions ^{whose} closed energy surfaces are all compact. Thus the quotient map $q : M \longrightarrow \Gamma$ will define a graph Γ , and all the local analysis used above will be applicable.

We will assume throughout this chapter that $H_1(M)$ is a finitely generated group.

5.1 Structure of Hamiltonian Systems

On a non-compact manifold M we define the sets $\mathcal{X}_N(M) = \{X_H \in \mathcal{X}_H(M) : X_H \text{ is a non-degenerate vector field on } M \text{ and all its closed energy surfaces are compact}\},$

$\mathcal{X}_S(M) = \{X_H \in \mathcal{X}_N(M) : \text{if } m \text{ is a critical point of } X_H \text{ then the closed energy surface containing } m \text{ contains precisely one singular point, viz. } m\}.$

Similarly define $\mathcal{H}_N(M)$ and $\mathcal{H}_S(M)$. $\mathcal{X}_N(M)$ and $\mathcal{X}_S(M)$ are called the set of non-degenerate Hamiltonian vector fields and the set of stable Hamiltonian vector fields resp.

Proposition 1 The sets $\mathcal{X}_N(M)$ and $\mathcal{X}_S(M)$ are open in $\mathcal{X}_H(M)$ with the C^r topology, $r \geq 1$.

Proof Transversality theory [7,p46]. ||

Note that $\mathcal{X}_N(M)$ and $\mathcal{X}_S(M)$ are no longer dense in $\mathcal{X}_H(M)$.

Theorem 2 Let $X_H \in \mathcal{X}_N(M)$. Let $K = \{x : x \text{ is a critical point of } H \text{ or } x \text{ is on a singular energy surface of } H\}$ and $L = \{x : x \text{ is on a regular energy surface of } H\} = M - K$. Then L is open and dense in M and L is a disjoint union of open annuli. ||

Remark We no longer require that L be a finite union of open annuli.

Corollary Each connected component of L is diffeomorphic to an open annulus $N \times \overset{\circ}{I}$ with energy surfaces $N \times \{t\}$, $t \in \overset{\circ}{I}$. ||

Proposition 3 The energy surfaces of $X_H \in \mathcal{X}_N(M)$ are consistent. ||

5.2 Hamiltonian Foliations

We define a Hamiltonian foliation of a non-compact manifold M to be a foliation with singularities of M satisfying

- 1) as section 3.2 (p83),
- 2) as section 3.2,
- 3) as section 3.2 except that C need not be finite,
- 4) as section 3.2,
- 5) as section 3.2 except that L may have infinitely many components,
- 6) as section 3.2,
- 7) every closed leaf of the foliation is compact.

Again the energy surfaces of a non-degenerate Hamiltonian function or vector field on M induce a Hamiltonian foliation of M , and the concept of \mathbb{E} -

equivalence extends naturally to Hamiltonian foliations on a non-compact manifold.

We use the same notation as in section 3.2 for the connected components of K and L except that now we may have infinitely many such components.

Let M be a manifold and F a Hamiltonian foliation of M . We construct a manifold w.b. M' as follows: Suppose that there exists a regular leaf E of F in M so that

a) cutting M along E separates M into two connected components M_1 and M_2 ,

b) M_2 , say, is diffeomorphic to $E \times [0,1)$ with foliation $E \times \{t\}$, $t \in [0,1)$.

Then replace M by M_1 . Continue this procedure until we can no longer find a leaf E satisfying (a) and (b), and call the resulting manifold M' and the foliation on it F' . Clearly $M \cong M' - \partial M'$ and we can find a diffeomorphism which preserves the foliations. Thus we can regard M as the interior of M' and F as the foliation induced by F' . Let K' and L' denote the decomposition of M' corresponding to K and L for M . Then $L \cong L'$, and there exists a diffeomorphism of K into K' so that the components of K' which are not in the image of K are precisely the boundary components of M' .

We assert that the map $q' : M' \longrightarrow \Gamma'$ defines a graph Γ' which we call the graph of F . We have a topological space Γ defined by the quotient map $q : M \longrightarrow \Gamma$ and a natural inclusion $\Gamma \subset \Gamma'$. $\Gamma' - \Gamma$ consists of a set of points corresponding to boundary components of M' . Also we have a map $q : M \longrightarrow \Gamma'$ which maps M into the graph Γ' of F .

As we will not need Γ below, and for compatability with our previous notation, we will in future denote the graph of F by $\Gamma = \Gamma(F)$.

Theorem 1 Let M be a connected manifold and F a Hamiltonian foliation on M . Then the graph $\Gamma = \Gamma(F)$ is a connected graph and

a) q induces a bijection from the set $\{\alpha_1, \dots\}$ indexing the components of L' to the set of edges of Γ .

b) q induces a bijection from the set $\{\beta_1, \dots\}$ indexing the components of K' to the set of vertices of Γ so that the edge corresponding to α_j is incident with the vertex corresponding to β_k if and only if $\bar{L}'_{\alpha_j} \cap \bar{K}'_{\beta_k} \neq \emptyset$.

Also the procedure described in section 3.3 for orienting Γ is applicable and orients Γ .

Proof Apply the arguments used in the compact case to the map $q' : M' \longrightarrow \Gamma'$.

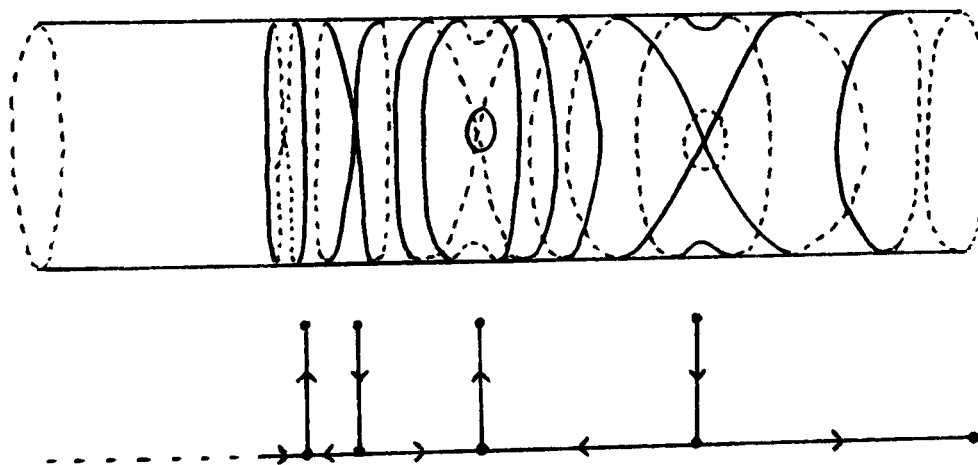
||

When we speak of a representative path for a path p in Γ we will always mean a representative path in M' . In general a representative path in M will not exist.

Proposition 2 If two Hamiltonian foliations of M are \mathbb{E} -equivalent then their oriented graphs are isomorphic.

||

Corollary If two Hamiltonian dynamical systems on M



A foliated manifold and its graph.

are \mathbb{E} -equivalent then their oriented graphs are isomorphic. ||

Proposition 3 If p is a finite path in Γ then there exists a representative path for p .

Proof We use the technique of Proposition 3.4.1 to generate a representative path for p . Since all closed energy surfaces are compact, and therefore contain at most a finite number of critical points in each, the technique will generate only a finite number of edges in the path p' corresponding to each edge of p . Hence p' is a finite path, and so the path P is a well defined path in M and a representative path for p . ||

Corollary If p is a closed path in Γ then we may choose a representative path P for p which is a closed path in M' .

Proof A closed path in Γ is necessarily finite. Hence a representative path P exists and, as in the Corollary to 3.4.1, we may choose a closed representative path P . ||

5.3 The C.S.I.

We define the C.S.I. of a Hamiltonian foliation \mathcal{F} as above (section 3.5) except that the sets X and D may now have infinitely many elements.

Theorem 1 Two Hamiltonian foliations \mathcal{F} and \mathcal{F}' on M are \mathbb{E} -equivalent if and only if they have the same C.S.I.

Proof The original proof (Theorem 3.5.6) is essentially local and consists of checking that the isotopy classes of the maps f_{α_j} are correct. Hence the proof applies equally when M is non-compact. ||

Theorem 3.5.7 is also applicable to the non-compact situation.

5.4 Applications to Hamiltonian Dynamics

All the results 3.6.1 to 3.6.5 and their proofs apply word for word to non-compact manifolds. To prove the equivalent of Proposition 3.6.6 we will need a preliminary lemma.

Lemma 1 Let M be a non-compact manifold and \mathcal{F} a Hamiltonian foliation of M . If $H_1(M)$ is finitely generated \checkmark then $\Gamma = \Gamma(\mathcal{F})$ is finitely connected. and $\Gamma(\mathcal{F})$ contains no oriented circuit

Proof Let β_i and β_j be two vertices of Γ . Since $H_1(M)$ is finitely generated there are at most finitely many paths P_k from K_{β_i} to K_{β_j} satisfying

a) the paths P_k are mutually non-homotopic,

b) on each path P_k the projection of P_k on Γ satisfies: if $t_1 < t_2$ then $q(P_k(t_1)) \leq q(P_k(t_2))$.
Now if p is an oriented path in Γ from β_i to β_j then there exists a representative path P for p , and P is homotopic to one of the P_k (look at the construction for P). Hence there are at most finitely many oriented paths p from β_i to β_j , for if p and p' are two such paths which are not identical then nor are they equivalent. But if they are not equivalent then they are not homotopic (keeping end points fixed) and so their representative paths are not homotopic.

Since any path from β_i to β_j is finite, we have proved that Γ is finitely connected. ||

The equivalent of Proposition 3.6.6 is

Proposition 2 Let F be a Hamiltonian foliation of M whose graph $\Gamma(F)$ contains no oriented circuit, and suppose that $H_1(M)$ is finitely generated. Then there exists a smooth function H on M inducing

the foliation F .

Proof The proof follows the same lines as Proposition 3.6.6 except that we use Proposition 2.8.2 in place of its corollary. ||

All the results 3.6.7 to 3.6.11 apply to non-compact manifolds when we add the additional hypothesis that $H_1(M)$ is finitely generated.

5.5 Stable Hamiltonian Systems

The results on stability of $X_H \in \mathcal{X}_S(M)$ are analagous to the results in chapter 3 , except that the stable Hamiltonian systems are no longer dense in $\mathcal{X}_H(M)$. The graph is of the special form, having all its vertices of types 1 to 5 .

Theorem 1 Let H be a stable Hamiltonian function on a manifold M . Then there exists a neighbourhood θ of H in $\mathcal{H}(M)$ with the C^2 topology such that $K \in \theta$ implies that H and K are \mathbb{E} -equivalent.

Proof Order the components of K as $K_{\beta_1}, K_{\beta_2}, \dots$ and let $X_{\beta_1}, X_{\beta_2}, \dots$ be canonical foliated neighbourhoods embedded in M and satisfying $\bigcup X_{\beta_i} = M$.

Put $U_j = \bigcup_{i=1}^j X_{\beta_i}$ and use the compact sets U_j to define the Whitney C^2 topology on M . Define a smooth function ϕ_j on U_j satisfying $\phi_j \equiv 0$ on a neighbourhood of ∂U_j , $\phi_j \equiv 1$ on $K \cap U_j$ and on any $L_{\alpha_i} \subset U_j$, and $\phi_j \in (0,1)$ elsewhere.

Now let H be a stable Hamiltonian function on M and K a C^2 close function. On U_j define a function K_j by $K_j(x) = \phi(x).K(x) + (1 - \phi(x))H(x)$. Then $K_j \equiv K$ except on a neighbourhood of ∂U_j and K_j is a Hamiltonian function on the compact manifold w.b. U_j . If K is sufficiently C^2 close to H on U_j then K and K_j have the same set of critical points, and H and K_j are \mathbb{E} -equivalent on U_j . The \mathbb{E} -equivalence of H with K_j maps energy surfaces of $H|_{U_j}$ onto energy surfaces of $K|_{U_j}$ except on a neighbourhood of ∂U_j .

But if H and K are sufficiently close in the Whitney C^2 topology on M then the last statement is true for every j , and so H and K are \mathbb{E} -equivalent on M . ||

Theorem 2 Let X_H be a stable Hamiltonian vector field on a manifold M . Then there exists a neighbourhood θ of X_H in $\mathcal{X}_H(M)$ with the C^1 topology such that $X_K \in \theta$ implies that X_H and X_K are \mathbb{E} -equivalent.

Proof The proof for vector fields is not an immediate corollary of Theorem 1 since the map \bar{X} is no longer an open map. But in fact we can use a proof precisely parallel to that of Theorem 1 to achieve the result. ||

5.6 The Cotangent Bundle

Let M be a manifold ($\partial M = \emptyset$). Recall that in chapter 2 we stated that the cotangent bundle T^*M has a natural symplectic structure. In this final section we investigate the \mathbb{E} -stability of an important class of functions on T^*M .

Let $V_0 : M \longrightarrow \mathbb{R}$ be a smooth function. V_0 induces a function $V : T^*M \longrightarrow \mathbb{R} : \alpha_m \longmapsto V_0(m)$.

Let g be a Riemannian metric on M , so that $g(m) : T_m M \times T_m M \longrightarrow \mathbb{R}$ is bilinear, symmetric, and positive definite for each $m \in M$. Define

$g_b : TM \longrightarrow T^*M$ by $g_b(m) : T_m M \longrightarrow T_m^* M$ and $g_b(m)(v_m) \cdot w_m = g(v_m, w_m)$ for all $w_m \in T_m M$. Then g_b

is a vector bundle isomorphism, and we denote the inverse isomorphism by $g_{\#}$. Define a function

$T : T^*M \longrightarrow \mathbb{R}$ by $T(\alpha_m) = \frac{1}{2} g(g_{\#}(\alpha_m), g_{\#}(\alpha_m))$.

We call V_0 a potential function and T a kinetic energy function.

$H = T + V$ is a Hamiltonian function on T^*M . Functions of this type are of great importance in Classical Mechanics, so we will investigate their qualitative properties. The first two results are well known.

Proposition 1 V_0 has a critical point at $m \in M$ if and only if H has a critical point at $0_m \in T^*M$.

Proof Take local coordinates $(q^1, \dots, q^n) = q$ on M and $(q^1, \dots, q^n, p_1, \dots, p_n) = (q, p)$ on T^*M . In local coordinates q is a critical point of V_0 if and only if $\frac{\partial V_0}{\partial q^1} = \dots = \frac{\partial V_0}{\partial q^n} = 0$. (q, p) is a critical point

of H if and only if

$$\frac{\partial H}{\partial q^1} = \dots = \frac{\partial H}{\partial q^n} = \frac{\partial H}{\partial p_1} = \dots = \frac{\partial H}{\partial p_n} = 0. \quad \text{In these}$$

coordinates $V = V(q)$ and $T = \frac{1}{2} g^{ij} p_i p_j$ (summation convention) where $g^{ij} = g^{ij}(q)$. Then

$$0 = \frac{\partial H}{\partial q^i} = \frac{\partial V}{\partial q^i} + \frac{1}{2} \frac{\partial g^{jk}}{\partial q^i} p_j p_k \quad \text{and} \quad 0 = \frac{\partial H}{\partial p_i} = g^{ij} p_j.$$

Since (g^{ij}) is positive definite everywhere, $\frac{\partial H}{\partial p_i} = 0$

if and only if $p = 0$. At $p = 0$ $\frac{\partial H}{\partial q^i} = \frac{\partial V}{\partial q^i} = 0$ if

and only if q is a critical point of V_0 . ||

Proposition 2 V_0 has a non-degenerate critical point at $m \in M$ if and only if H has a non-degenerate critical point at $0_m \in T^*M$.

Proof In local coordinates q on M and (q,p) on T^*M , a critical point of V_0 at q is non-degenerate if and only if

$$\det \left(\frac{\partial^2 V_0}{\partial q^i \partial q^j} \right) \neq 0 ,$$

and a critical point of H at (q,p) is non-degenerate if and only if

$$\det \begin{pmatrix} \frac{\partial^2 H}{\partial q^i \partial q^j} & \frac{\partial^2 H}{\partial q^i \partial p_j} \\ \frac{\partial^2 H}{\partial p_i \partial q^j} & \frac{\partial^2 H}{\partial p_i \partial p_j} \end{pmatrix} \neq 0$$

But this second determinant is equal to

$$\det \begin{pmatrix} \frac{\partial^2 V_0}{\partial q^i \partial q^j} & 0 \\ 0 & g^{ij} \end{pmatrix} = \det \left(\frac{\partial^2 V_0}{\partial q^i \partial q^j} \right) \cdot \det (g^{ij})$$

since all critical points occur on the zero section. But since $\det (g^{ij}) \neq 0$ everywhere, the product is zero if and only if

$$\det \left(\frac{\partial^2 V_0}{\partial q^i \partial q^j} \right) \neq 0 . \quad \parallel$$

We now look at stability properties of Hamiltonian functions $H = T + V$.

Theorem 3 $H = T + V$ is a stable Hamiltonian function on T^*M if and only if

a) V_0 is a stable Hamiltonian function on M ,
 b) whenever m and m' are critical points of V_0 having the same critical value e , then every path in M from m to m' contains a point p such that $V_0(p) > e$,

c) each connected component U of the set $U_k = \{m \in M : V_0(m) < k, k \in \mathbb{R}\}$ is contained in a compact subset of M .

Proof \rightarrow . Suppose that H is a stable Hamiltonian function. If we identify M with the zero section of T^*M then $V_0 = H|_M = V|_M$. Now H is a stable Hamiltonian function if and only if all closed energy surfaces of H are compact, all critical points of H are non-degenerate, and if m is a critical point of H then the closed energy surface containing m contains no other critical point.

a) By Propositions 1 and 2, all critical points of V_0 are non-degenerate. Since energy surfaces of V_0 are energy surfaces of H intersected with the zero section of T^*M , all energy surfaces of V_0 are compact and if m is a critical point of V_0 then the closed energy surface containing m contains no other critical point. Hence V_0 is stable.

b) Suppose that m, m' are two critical points of V_0 with critical value e , and P is a path in M from m to m' . If there does not exist a point p in P with $V_0(p) > e$, then for all $p \in P$ we have $V_0(p) \leq e$. Thus there exists $\alpha_p \in T^*_p M$ such that $H(\alpha_p) = e$, since T is positive definite. Thus there exists a path $P' \subset H^{-1}(e) \subset T^*M$ from 0_m to $\alpha_{m'}$. But $T(\alpha_{m'}) = H(\alpha_{m'}) - V(\alpha_{m'}) = H(\alpha_{m'}) - H(0_{m'}) = e - e = 0$, so $\alpha_{m'} = 0_{m'}$ and $0_m, 0_{m'}$ are in the same connected component of $H^{-1}(e)$. This contradicts the stability of H .

c) Suppose that (c) is false. Then there exists a constant k such that U_k has a connected component U which is not contained in any compact set. U is open and therefore a submanifold of M . Let $T^*U \subset T^*M$ be the restriction of T^*M to U . Then for all $K > k$ $H^{-1}(K) \cap T^*U$ is a sphere bundle over U . If $H^{-1}(K) \cap T^*U$ is contained in a closed energy surface E which is compact then the projection of E on M is a compact set containing U . But this last statement contradicts either the stability of H or the hypothesis that (c) is false. Therefore (c) is true.

<— . First, all critical points of H are non-degenerate since those of V_0 are.

Let E be a closed energy surface of H with

energy e . Let E' be the projection of E on M . Then if $m \in E'$, $V_0(m) \leq e$. By hypothesis (c) each connected component of U_{e+1} is contained in a compact subset of M . Let U be the connected component of U_{e+1} containing E' and W a compact subset of M containing U . On W V_0 has a minimum value $a \leq e$. Thus E is a closed subset of the compact set $\{\alpha_m \in T^*M : m \in W \text{ and } T(\alpha_m) \leq e - a + 1\}$ which is compact.

Suppose that E is a closed energy surface of H containing two distinct critical points m and m' . Let P be a path in E from m to m' , and let P' be the projection of P onto M . Then at every point $p \in P'$ we have $V_0(p) \leq e$. But this contradicts (b). Hence a closed energy surface of H contains at most one critical point.

Thus H is a stable Hamiltonian function. ||

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Morse-Smale type Inequalities for
Hamiltonian Dynamical Systems

by

A.R. Walker

Introduction

In this paper we deduce some Morse-Smale type inequalities for Hamiltonian Dynamics. We will classify the critical points of a Hamiltonian dynamical system according to the characteristic exponents at those points. Although there is as yet no complete Hartman type characterisation of the flow at such a critical point, we do have a partial result in this direction (the Arnold-Liapounov-Kelley subcentre theorem [1]) which we think is sufficiently powerful to justify a study of such critical points.

We then adapt the Morse Inequalities [2] to get inequalities for Hamiltonian dynamical systems in terms of the critical points classified by characteristic exponents.

The basic results which we will use in this paper are to be found in references [1] to [4]. In particular we will use the notations of the previous paper.[4].

The Morse Inequalities

Let H_* be the singular homology functor (coefficients in \mathbb{Z}). Let M^n be a manifold w.b. and $H \in \mathcal{H}_N(M)$. If e is a non-critical value of H define

$M_e = \{x \in M : H(x) \leq e\}$. Then M_e is a submanifold w.b. of M . Let $a < b$, a, b non-critical values of H , and suppose that H has finitely many critical points in $H^{-1}[a, b]$. Define

$$R_q(M_b, M_a) = \dim H_q(M_b, M_a) = q \text{ th Betti number of } (M_b, M_a),$$

$$C_q(M_b, M_a) = \text{number of critical points of } H \text{ in } H^{-1}[a, b] \text{ with Morse index } q,$$

$$\chi(M_b, M_a) = \sum_{q=0}^n (-1)^q R_q(M_b, M_a) = \text{Euler characteristic of } (M_b, M_a).$$

Then for the two cases in which we are interested we have:

Morse Inequalities Let M^n be a manifold w.b. and $H \in \mathcal{H}_N(M)$. Let $a < b$, a, b non-critical values of H , and suppose that H has finitely many critical points in $H^{-1}[a, b]$. Putting $R_q = R_q(M_b, M_a)$, $C_q = C_q(M_b, M_a)$, and $\chi = \chi(M_b, M_a)$ we have

$$\sum_{q=0}^k (-1)^{k-q} R_q \leq \sum_{q=0}^k (-1)^{k-q} C_q, \quad k = 0, \dots, n-1.$$

and

$$\chi = \sum_{q=0}^n (-1)^q R_q = \sum_{q=0}^n (-1)^q C_q.$$

Proof [2].

||

Corollary 1 Let M^n be a compact manifold w.b., and

$H \in \mathcal{H}_N(M)$. Let ∂M_- be the subset of ∂M on which H has a local minimum. Then putting $R_q = R_q(M, \partial M_-)$, $C_q = C_q(M, \partial M_-)$, and $\chi = \chi(M, \partial M_-)$ we have

$$\sum_{q=0}^k (-1)^{k-q} R_q \leq \sum_{q=0}^k (-1)^{k-q} C_q, \quad k = 0, \dots, n-1.$$

$$\chi = \sum_{q=0}^n (-1)^q R_q = \sum_{q=0}^n (-1)^q C_q. \quad ||$$

Corollary 2 In both the above cases we have $R_q \leq C_q$, for $q = 0, \dots, n$. ||

Let $\rho_q(M_b, M_a) = \rho_q$ be the number of generators of the torsion of $H_q(M_b, M_a)$. Then we can strengthen the Morse Inequalities as follows [5],[6].

Strong Morse Inequalities With the hypotheses of the Morse Inequalities or of the first corollary we have

$$\sum_{q=0}^k (-1)^{k-q} R_q + \rho_k \leq \sum_{q=0}^k (-1)^{k-q} C_q,$$

$k = 0, \dots, n-1$, and $R_q + \rho_q + \rho_{q-1} \leq C_q$, $q = 0, \dots, n$.
(Note that $\rho_{-1} = \rho_0 = \rho_n = 0$.) ||

Behaviour near a Critical Point

We look at the behaviour of a Hamiltonian vector field or function at its critical points. Recall that the set of critical points of $H \in \mathcal{H}(M)$ is identical with the set of critical points of $X_H \in \mathcal{X}_H(M)$.

Let $0_m \in T_m M$ be the zero vector. Then $T_{0_m}(TM)$ has two distinguished subspaces, $T_{0_m}(T_m M)$ and $T_{0_m}(0_{TM})$, the 'vertical' and 'horizontal' tangent spaces, each of which has a natural identification with $T_m M$. Let $v_m : T_{0_m}(TM) \longrightarrow T_{0_m}(T_m M) \longrightarrow T_m M$ be the natural projection on the vertical tangent space.

If m is a critical point of $X \in \mathcal{X}(M)$, let $X'(m) = v_m \circ T_m X \in L(T_m M, T_m M)$. The characteristic exponents of X at m are the eigenvalues of $X'(m)$. The characteristic multipliers of X at m are the exponentials of the characteristic exponents. In local coordinates (x^1, \dots, x^n) , if $X(m) = (X^1(m), \dots, X^n(m))$ then $X'(m)$ is the matrix

$$\left[\frac{\partial X^i}{\partial x^j}(m) \right] .$$

The characteristic exponents of $X_H \in \mathcal{X}_H(M)$ at a critical point $m \in M$ occur in pairs $(\lambda, -\lambda)$ of the same multiplicity. Thus if λ is a characteristic exponent then so are $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$, all with the same multiplicity. 0 always has even multiplicity.

At each critical point $m \in M^{2n}$ of X_H we have $2n$ characteristic exponents in pairs $(\lambda, -\lambda)$. With the critical point m we associate a triple (a, b, c) where a, b, c , are non-negative integers, a is the number of pairs $(\lambda, -\lambda)$ with $\operatorname{Re}(\lambda) \neq 0$, b is the number of

pairs for which $\operatorname{Re}(\lambda) = 0$, $\operatorname{Im}(\lambda) \neq 0$, and c is the number of zero pairs. (a,b,c) is called the Hamiltonian type of the critical point m , and a , b , c , the saddle type, centre type, and zero type, resp., for obvious reasons. Clearly $a + b + c = n$.

If $c = 0$ then the critical point m is non-degenerate. If, in addition, all the characteristic exponents are distinct, we say that m is a strongly non-degenerate critical point. If all critical points of X_H are strongly non-degenerate then we say that X_H is a strongly non-degenerate vector field.

The following result is well known:

Theorem 1 [3]. Suppose $X_H \in \mathcal{X}_H(M)$ and $m \in M$ is a strongly non-degenerate critical point of X_H . Suppose that there exist symplectic coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ in a neighbourhood of m w.r.t. which H is a homogeneous quadratic function in q^1, \dots, p_n . Then there exists a linear transformation $(q,p) \rightsquigarrow (x,y)$ which is (formally) symplectic and reduces H to the form

$$H(x,y) = H(0,0) + \sum_{i=1}^n \lambda_i x^i y_i$$

where $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$ are the distinct characteristic exponents. ||

Unfortunately, to achieve this result we must allow

x^i and y_i to be complex valued coordinates. We can improve on this result as follows:

Suppose that we arrange the characteristic exponents so that

$$(\lambda_1, \dots, \lambda_n) = (ia_1, \dots, ia_r, \beta_1, \dots, \beta_s, \gamma_1 + i\delta_1, \gamma_1 - i\delta_1, \dots, \gamma_t - i\delta_t)$$

where $r + s + 2t = n$, a_i , β_j , γ_k , and δ_l are all real and non-zero, and the $2n$ characteristic exponents are $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$.

Theorem 2 With the above hypothesis and the hypotheses of Theorem 1, there exists a real linear transformation $(q, p) \rightsquigarrow (x, y)$ which is symplectic and reduces H to the form

$$\begin{aligned} H(x, y) = & H(0, 0) + \sum_{i=1}^r \frac{1}{2} a_i (x_i^2 + y_i^2) + \sum_{j=1}^s \beta_j x^{r+j} y_{r+j} \\ & + \sum_{k=1}^t \left[\gamma_k (x^{u+2k-1} y_{u+2k-1} + x^{u+2k} y_{u+2k}) \right. \\ & \left. + \delta_k (x^{u+2k-1} y_{u+2k} - x^{u+2k} y_{u+2k-1}) \right] \end{aligned}$$

where $u = r + s$.

Remark This theorem also appears to be well known, at least in simple cases (when there are no complex exponents for example), but as we have not seen a proof in the literature we will give one below. The proof uses Pars

proof of Theorem 1 , choosing eigenvectors carefully so that when we transform from the canonical form of Theorem 1 to the canonical form of Theorem 2 , the resulting composite transformation is real.

Proof Replacing $q^1, \dots, q^n, p_1, \dots, p_n$ by q^1, \dots, q^{2n} we can write H in the form

$H(q) = \frac{1}{2} q' S q$ where S is a real symmetric $2n \times 2n$ matrix. As in Pars proof we first find a transformation K such that

$$K' Z K = Z \quad \text{and} \quad K' S K = E$$

where $Z = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix}$,

$$L = \text{diag}(i\alpha_1, \dots, \gamma_t - i\delta_t) = \text{diag}(\lambda_1, \dots, \lambda_n) .$$

The characteristic exponents are the eigenvalues of the matrix $A = ZS$, i.e. the roots of the equation $f(\lambda) = 0$, where

$$f(\lambda) = |A - \lambda 1_{2n}| = |ZS - \lambda 1_{2n}| .$$

Now $Z(S + \lambda Z) = ZS - \lambda 1_{2n}$, and since $|Z| = 1$,

$$f(\lambda) = |S + \lambda Z| .$$

Also $(S + \lambda Z)' = S - \lambda Z$, so

$$f(\lambda) = |S - \lambda Z| .$$

Thus (as we have already assumed) we may take the eigenvalues to be $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$, ordered as we have supposed.

In C^{2n} there exist $2n$ linearly independent eigenvectors c_1, \dots, c_{2n} , and we may choose them to satisfy

$$\begin{aligned} c_i &= \bar{c}_{n+i}, \text{ for } i = 1, \dots, r, \\ c_{r+1}, \dots, c_{r+s}, c_{n+r+1}, \dots, c_{n+r+s} &\text{ real,} \\ \left. \begin{aligned} c_{u+2k-1} &= \bar{c}_{u+2k} \\ c_{v+2k-1} &= \bar{c}_{v+2k} \end{aligned} \right\} & k = 1, \dots, t, \text{ and} \\ & u = r+s, v = n+r+s. \end{aligned} \quad (1)$$

Since $\lambda_h = -\lambda_{n+h}$, $h = 1, \dots, n$, these eigenvectors satisfy

$$Ac_h = \lambda_h c_h, \quad Ac_{n+h} = -\lambda_h c_{n+h}$$

and

(2)

$$Sc_h = -\lambda_h Zc_h, \quad Sc_{n+h} = \lambda_h Zc_{n+h}$$

Let B be the $2n \times 2n$ matrix given by

$$b_{ij} = c_i' S c_j. \text{ If } g, h \leq n \text{ we have}$$

$$\begin{aligned} b_{gh} &= c_g' S c_h = -\lambda_h c_g' Z c_h \\ &= c_h' S c_g = -\lambda_g c_h' Z c_g = \lambda_g c_g' Z c_h \end{aligned}$$

and since $\lambda_h \neq -\lambda_g$, $b_{gh} = 0$. Similarly $b_{n+g, n+h} = 0$.

$$\begin{aligned} b_{g, n+h} &= \lambda_h c_g' Z c_{n+h} \\ &= \lambda_g c_g' Z c_{n+h} \end{aligned}$$

so $b_{g, n+h} = 0$ if $g \neq h$. Similarly $b_{n+g, h} = 0$ if $g \neq h$. Since S is symmetric $b_{g, n+h} = b_{n+h, g}$.

Let $d_h = c_h' Z c_{n+h}$. Then $d_h \neq 0$. For let C be the matrix $(c_1 \dots c_{2n})$. C is non-singular and so is $C' Z C$. But $C' Z C$ is the matrix $(c_i' Z c_j)$ which is equal to

$\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$ where $D = \text{diag}(d_1, \dots, d_n)$. Hence none of

d_1, \dots, d_n is zero.

We can now form new eigenvectors e_1, \dots, e_{2n} for which $e'_h Z e_{n+h} = 1$, and all other terms $e'_g Z e_h$ are zero.

If $1 \leq l \leq r$ we have $d_l = c'_1 Z \bar{c}_1$
 $= \sum_{j=1}^n (c_{1j} \cdot \bar{c}_{1,j+n} - c_{1,j+n} \cdot \bar{c}_{1j}) = \alpha_1 - \bar{\alpha}_1 = -iy_1$ for
 some real number y_1 . W.l.o.g. we may assume $y_1 > 0$,
 for if not interchange c_1 and c_{n+1} . Put $e_1 = y_1^{-\frac{1}{2}} c_1$
 and $e_{n+1} = iy_1^{-\frac{1}{2}} c_{n+1}$.

If $r < j \leq r+s$ we have $d_j = c'_j Z c_{n+j}$ is real
 and again (w.l.o.g.) $d_j > 0$. Put $e_j = \frac{1}{d_j} c_j$ and
 $e_{n+j} = c_{n+j}$.

If $1 \leq k \leq t$ we have

$$d_{u+2k-1} = c'_{u+2k-1} Z c_{v+2k-1},$$

$$\begin{aligned} d_{u+2k} &= c'_{u+2k} Z c_{v+2k} = \bar{c}'_{u+2k-1} Z \bar{c}_{v+2k-1} \\ &= \bar{d}_{u+2k-1} \end{aligned}$$

Put $e_{u+2k-1} = (d_{u+2k-1})^{-1} c_{u+2k-1}$, $e_{u+2k} = (d_{u+2k})^{-1} c_{u+2k}$,
 $e_{v+2k-1} = c_{v+2k-1}$, $e_{v+2k} = c_{v+2k}$.

Then our new eigenvectors e_1, \dots, e_{2n} satisfy

$$e_1 = i\bar{e}_{n+1}, \text{ for } l = 1, \dots, r,$$

$$e_{r+1}, \dots, e_{r+s}, e_{n+r+1}, \dots, e_{n+r+s} \text{ real,}$$

$$\left. \begin{aligned} e_{u+2k-1} &= \bar{e}_{u+2k} \\ e_{v+2k-1} &= \bar{e}_{v+2k} \end{aligned} \right\} \begin{aligned} &k = 1, \dots, t, \text{ and} \\ &u = r+s, v = n+r+s. \end{aligned}$$

and equations (2) .

Hence if we let K be the matrix $(e_1 \dots e_{2n})$ then

$$K' Z K = Z \quad \text{and} \quad K' S K = E .$$

Thus using the matrix K to make a formally symplectic coordinate change we have H in the form

$$H(q,p) = \sum_{i=1}^n \lambda_i q^i p_i .$$

We now make a second symplectic coordinate change, putting

$$\begin{aligned} \sqrt{2} \cdot q^1 &= y_1 - ix^1, \quad \sqrt{2} \cdot p_1 = iy_1 - x^1, \quad l = 1, \dots, r, \\ q^1 &= x^1, \quad p_1 = y_1, \quad l = r+1, \dots, r+s, \\ \sqrt{2} \cdot q^{u+2k-1} &= x^{u+2k-1} + ix^{u+2k} \\ \sqrt{2} \cdot q^{u+2k} &= x^{u+2k-1} - ix^{u+2k} \\ \sqrt{2} \cdot p_{u+2k-1} &= y_{u+2k-1} - iy_{u+2k} \\ \sqrt{2} \cdot p_{u+2k} &= y_{u+2k-1} + iy_{u+2k}, \quad k = 1, \dots, t. \end{aligned}$$

This coordinate change is again symplectic and takes H to the required form. We leave it to the reader to check this assertion. It remains to prove that the composite coordinate change is real. This second transformation has matrix

$$\begin{pmatrix} -i1_r & 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_s & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{2t} & 0 & 0 & 0 \\ -1_r & 0 & 0 & i1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_s & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{J}_{2t} \end{pmatrix}$$

where J_{2t} is the $2t \times 2t$ matrix

$$\begin{pmatrix} 1 & i & 0 & 0 & 0 & \dots \\ 1 & -i & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & i & 0 & \dots \\ 0 & 0 & 1 & -i & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Since $K = (e_1 \dots e_{2n})$, KL is the matrix $(f_1 \dots f_{2n})$ where

for $1 \leq j \leq r$, $f_j = -ie_j - e_{n+j} = -ie_j + i\bar{e}_j$
and $f_{n+j} = e_j + ie_{n+j} = e_j + \bar{e}_j$. Hence f_j and f_{n+j}
are real vectors,

for $r < j \leq r+s$, $f_j = e_j$, and $f_{n+j} = e_{n+j}$.
Hence f_j and f_{n+j} are real vectors.

for $1 \leq k \leq t$, $f_{u+2k-1} = e_{u+2k-1} + e_{u+2k}$,
 $f_{u+2k} = ie_{u+2k-1} - ie_{u+2k}$.
Hence f_{u+2k-1} and f_{u+2k} are real vectors. Similarly
 f_{v+2k-1} and f_{v+2k} are real vectors.

Thus KL is a real symplectic matrix, and the
result follows. ||

Now suppose that $H \in \mathcal{H}(M)$ is a Hamiltonian
function with critical point at m . Then choosing
symplectic coordinates at m and using Taylor's Theorem
we can write H in the form $H = H_2 + H'$ in a

neighbourhood of m , where H_2 is a homogeneous quadratic function of the symplectic coordinates and $DH'(0) = 0$, $D^2H'(0) = 0$. Note that the characteristic exponents of H depend only on H_2 . Hence:

Corollary Suppose that $H \in \mathcal{H}(M)$ is a Hamiltonian function with strongly non-degenerate critical point at $m \in M$. Then there exists a symplectic chart in a neighbourhood of m w.r.t. which H takes the form

$$H = H_2 + H'$$

where H_2 is a quadratic homogeneous function of the symplectic coordinates in canonical form (Theorem 2), and $DH'(0) = 0$, $D^2H'(0) = 0$. ||

Density Properties

On a manifold w.b. M , define the set

$$\mathcal{X}_D(M) = \{X_H \in \mathcal{X}_H(M) : X_H \text{ is a strongly non-degenerate vector field whose closed energy surfaces are compact}\}.$$

Similarly $\mathcal{H}_D(M)$.

Proposition 3 If M is compact then $\mathcal{X}_D(M)$ is open dense in $\mathcal{X}_H(M)$. If M is non-compact then $\mathcal{X}_D(M)$ is open in $\mathcal{X}_H(M)$.

Proof Transversality theory. ||

Number of Critical Points

Proposition 4 Let $X_H \in \mathcal{X}_H(M)$ and suppose that $m \in M$ is a strongly non-degenerate critical point of H of saddle type a . Then the Morse index of H at m is $a + 2q$ where $0 \leq q \leq n-a$.

Proof There exist symplectic coordinates in a neighbourhood of m w.r.t. which $m = 0$, and H has the form

$$H = H_2 + H'$$

where H_2 is a homogeneous quadratic function of the symplectic coordinates in canonical form (Theorem 2) and $DH'(0) = 0$, $D^2H'(0) = 0$. Thus the Morse index at m is determined by H_2 .

At m we have $2(n-a)$ characteristic exponents $(i\alpha, -i\alpha)$ which are purely imaginary. The canonical form corresponding to each such pair is $\frac{1}{2} \alpha(x^2 + y^2)$ whose contribution to the Morse index is 0 or 2 according as α is positive or negative.

The remaining $2a$ characteristic exponents are pairs $(\beta, -\beta)$ with β real, or quadruples $(\gamma+i\delta, \gamma-i\delta, -\gamma-i\delta, -\gamma+i\delta)$, γ, δ , real. Corresponding to each pair $(\beta, -\beta)$ we have canonical form $\beta xy (= \beta(u^2 - v^2))$ with the obvious coordinate change). Thus the contribution to the Morse index is precisely 1. Corresponding to each

quadruple $(\gamma + i\delta, \dots)$ we have canonical form

$\gamma(x_1 y_1 + x_2 y_2) + \delta(x_1 y_2 - x_2 y_1)$ ($= x_1 u + x_2 v = u_1^2 - v_1^2 + u_2^2 - v_2^2$ with the obvious coordinate changes). Thus contribution to the Morse index is 2.

Hence the Morse index at m is $a + 2q$ where $0 \leq q \leq n-a$. ||

Let $X_H \in \mathcal{X}_D(M)$ and let $S_\lambda = S_\lambda(M_b, M_a)$ be the number of critical points of X_H of saddle type λ in $H^{-1}[a, b]$.

Theorem 5 Let $X_H \in \mathcal{X}_D(M)$ be a vector field on M^{2n} .

With the above notations we have

$$\chi(M_b, M_a) = \sum_{q=0}^{2n} (-1)^q R_q = \sum_{q=0}^n (-1)^q S_q$$

and

$$\sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} (R_{\lambda-2q} + R_{2n-\lambda+2q}) + \sum_{q=0}^{\lambda+1} (\rho_{\lambda-q} + \rho_{2n-\lambda-1+q}) - \delta_\lambda^n (R_n + \rho_n + \rho_{n-1}) \leq \sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} S_{\lambda-2q}, \quad \lambda = 0, \dots, n.$$

In particular

$$\sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} (R_{\lambda-2q} + R_{2n-\lambda+2q}) - \delta_\lambda^n R_n \leq \sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} S_{\lambda-2q}$$

for $\lambda = 0, \dots, n$.

Proof For the first part we have the Morse inequality

$$\chi = C_0 - C_1 + \dots + C_{2n},$$

and by Proposition 4 we have

$$C_0 + C_2 + \dots + C_{2n} = S_0 + S_2 + \dots + S_{2\lfloor \frac{1}{2}n \rfloor}$$

and $C_1 + C_3 + \dots + C_{2n-1} = S_1 + S_3 + \dots + S_{2[\frac{1}{2}n+\frac{1}{2}]-1}$

Combining the last two equations gives the required result.

The third proposition is a direct consequence of the second. To prove the second we use the corollary to the Morse inequalities, $R_q + \rho_q + \rho_{q-1} \leq C_q$. In addition

$$\sum_{q=0}^{[\frac{1}{2}\lambda]} (C_{\lambda-2q} + C_{2n-\lambda+2q}) - \delta_{\lambda}^n C_n \leq \sum_{q=0}^{[\frac{1}{2}\lambda]} S_{\lambda-2q}.$$

For suppose that m is a critical point of X_H of saddle type a . Then its Morse index is $a+2q$ where $0 \leq q \leq n - a$. Hence if $a > \lambda$, then $\lambda < a + 2q < 2n - \lambda$, so if the critical point m is not counted by the sum on the r.h.s., then it is not counted by the sum on the l.h.s. Thus the inequality holds, and combining it with the previous inequality gives the result. \parallel

Theorem 6 Let $X_H \in \mathcal{X}_D(M)$ be a vector field on M^{2n} with finitely many critical points. Put $S_q = S_q(M, \partial M_-)$, etc. Then

$$\chi(M, M_-) = \sum_{q=0}^{2n} (-1)^q R_q = \sum_{q=0}^n (-1)^q S_q$$

and

$$\begin{aligned} \sum_{q=0}^{[\frac{1}{2}\lambda]} (R_{\lambda-2q} + R_{2n-\lambda+2q}) + \sum_{q=0}^{\lambda+1} (\rho_{\lambda-q} + \rho_{2n-\lambda+q}) \\ - \delta_{\lambda}^n (R_n + \rho_n + \rho_{n-1}) \leq \sum_{q=0}^{[\frac{1}{2}\lambda]} S_{\lambda-2q}, \quad \lambda = 0, \dots, n. \end{aligned}$$

In particular

$$\sum_{q=0}^{[\frac{1}{2}\lambda]} (R_{\lambda-2q} + R_{2n-\lambda+2q}) - \delta_{\lambda}^n R_n \leq \sum_{q=0}^{[\frac{1}{2}\lambda]} S_{\lambda-2q}$$

for $\lambda = 0, \dots, n$. \parallel

In fact we can strengthen these inequalities slightly. Let $S_{\lambda\mu}$ denote the number of critical points of $X_H \in \mathcal{K}_D(M)$ with saddle type λ and Morse index μ where $\mu \leq \lambda$ or $\mu \geq 2n - \lambda$, $\mu \leq n$. Note that $S_{\lambda\mu} \leq S_\lambda$ and that $S_{\lambda\mu} = 0$ if $\mu < \lambda$.

Corollary 1 With the hypotheses of Theorem 5 or Theorem 6, we have

$$\begin{aligned} \sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} (R_{\lambda-2q} + R_{2n-\lambda+2q}) + \sum_{q=0}^{\lambda+1} (\rho_{\lambda-q} + \rho_{2n-\lambda-1+q}) \\ - \delta_\lambda^n (R_n + \rho_n + \rho_{n-1}) \leq \sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} S_{\lambda-2q,\lambda}, \text{ and} \\ \sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} (R_{\lambda-2q} + R_{2n-\lambda+2q}) - \delta_\lambda^n R_n \leq \sum_{q=0}^{\lfloor \frac{1}{2}\lambda \rfloor} S_{\lambda-2q,\lambda} \end{aligned}$$

for $\lambda = 0, \dots, n$.

Proof Precisely the same argument is valid. ||

However these inequalities are still very weak since they give no estimates for S_λ or $S_{\lambda\mu}$ individually.

Corollary 2 If M^{2n} is a closed manifold then putting $S = S(M)$, etc., we have

$$\sum_{q=0}^{\frac{1}{2}\lambda} S_{\lambda-2q} \geq \sum_{q=0}^{\frac{1}{2}\lambda} S_{\lambda-2q,\lambda} \geq \lambda + p,$$

where λ is even, $0 \leq \lambda \leq n$, and $p = 1$ if $\lambda = n$, $p = 2$ otherwise.

Proof Recall that if M is a closed symplectic manifold then none of the even dimensional homology groups is trivial. ||

Corollary 3 Let M be a symplectic 2-manifold w.b. and let $X_H \in \mathcal{X}_D(M)$. Then S_0 is the number of centres of X_H , S_1 is the number of saddles of X_H , and if ∂M has r connected components then

$$\chi(M, \partial M_-) = S_0 - S_1 ,$$

$$S_0 \geq \max(0, 2-r) ,$$

$$S_1 \geq -\chi(M, \partial M_-) + \max(0, 2-r) .$$
||

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